A SECOND REPORT
ON THE
TEACHING OF GEOMETRY IN SCHOOLS

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PREPARED FOR THE
MATHEMATICAL ASSOCIATION

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INSTITUTE OF EDUCATION FOR WOMEN
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THIS report was prepared by the Boys' Schools Committee of the Mathematical Association appointed in December, 1933. It was adopted by the General Teaching Committee at a meeting held on 16th October, 1937.

The Boys' Schools Committee was composed as follows :

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For brevity the language used in the report refers to ' boys ' only. The Committee is unanimously of the opinion that in geometry no distinction should be made between a course for boys and a course for girls, except in so far as certain types of illustration or rider will be less appropriate to one sex than to the other.

In preparing this report the Committee has had a good deal of assistance from members of the Girls' Schools Committee, and considerable modifications, notably as regards Stage A, have been due to their initiative.

Note added in the 1951 issue. The Mathematical Gazette XXIII (1939) contains (pp. 169-185) a full report of a discussion on this Second Geometry Report at the Annual General Meeting in 1939, and (p. 185-197) an article on this Report by Dr. G. Wolff of Düsseldorf.

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I INTRODUCTION

The Connection of this Report with that of 1923. In 1923 the Mathematical Association issued a report on *The Teaching of Geometry in Schools*. This has been a ' best seller ' among the reports issued by the Association ever since, and the present report is to be regarded as a companion to it.

Most of the present report (excluding the appendix) is an amplification of chapters I and II of the older one, an amplification which goes into very much greater detail both as to the subject matter to be taught and the methods of teaching. The only serious difference of opinion here expressed is with regard to the later part of the course, Stage C, the systematising stage. Of this the older report said ' there is little experience to guide one in making recommendations as to this stage '. Since 1923 a good deal of experience of this part of the course has been gained and in the present report definite suggestions about it are made which to a considerable extent conflict with and modify those of the older report.

There are one or two minor differences, but on the whole the teacher who reads chapters I and II of the older report *pari passu* with chapters II to IX of this one, will find that what is said here elucidates and amplifies them.

The main bulk (about two-thirds) of the 1923 Report, namely chapters III to VII, is concerned with questions which are not dealt with at all in the present one, while the appendix to this report contains new matter not dealt with in the other.

The suggestions in this report are based on the existing state of affairs in which elementary geometry is largely regarded as an isolated subject of the curriculum with a separate technique of its own. This is not to be taken to imply hostility to a possible closer fusion of mathematical subjects. In such a fusion it is not unlikely that the strictly geometrical part of the course would be lightened by the introduction of trigonometry in somewhat the same way as it has already been lightened by the use of algebra.

The Division of the Geometry Course into Stages. One of the most valuable features of the 1923 Report is the division of the ordinary course of deductive geometry into two stages, Stage B and Stage C, the deductive and systematising stages (Stage A being experimental geometry).

The distinction may perhaps be put briefly thus (but see p. 15 of the 1923 Report). In the ' deductive ' stage the interest is in the search for geometrical truths while in the ' systematising ' stage much of it is transferred to the way in which these truths are linked together.

Much hard thinking and experimenting have been given since 1923 to the distinction and connection between these stages and as a result the present report puts forward suggestions modifying as well as amplifying the older ones.

In the 1923 Report, the systematising stage is described as consisting of two parts, ' the organisation of derived propositions ' and ' the organisation of primitive propositions '. Experience has shown that the second of these is suitable only for top divisions and not for the majority of boys. It is therefore here relegated to a secondary position as Stage C *b*.

The organisation of derived propositions is called Stage C *a* and is desirable for all. This may also be called the ' Consolidating Stage '. Detailed recommendations as to this work are given in chapter VIII.

Contents of the 1923 Report. It seems well to give here some indication of the contents of the 1923 Report, for the benefit of any teacher who may not yet have read it.

Chapters I and II are headed ' General Principles ' and ' Elementary Practical Conclusions '. Most of the contents of these chapters are treated in greater detail in the present report. The important difference in what is recommended for Stage C has already been stated.

Chapter III, ' Discussion of some disputed points ', is the ' longest and in some ways the most important part ' of the 1923 Report, and its contents are not touched in the present one.

It provides ' something that to most teachers has hitherto been inaccessible, namely, a careful discussion of the principal difficulties encountered in the attempt to keep a course in geometry free from arguments that even the commonsense logic which is appropriate to the schoolroom must recognise as defective '.

The lapse of time since 1923 has not diminished the value of this discussion.

The remaining chapters are as follows :

Chapter IV	The Question of an Agreed Sequence. Difficulties in Teaching and Examining.
Chapter V	Notes on some minor points.

Chapter VI A word with the examiner.
Chapter VII A note on Relativity.

A caution may be added here about a part of the Report which led to a certain amount of misunderstanding, viz. that part which discussed the possibility of replacing the parallel postulate by the principle of similarity.

The implications of this proposal were then (as also they are now) unfamiliar to most teachers and it was therefore necessary in order to avoid obscurity to go into considerable detail. The mere bulk of this detail gave, especially to the careless reader, the impression that one of the main objects of the report was to *advocate* this change from one fundamental postulate to another, and led to a good many criticisms of the report which would not have been made had the critics confined themselves to the actual recommendation.

The recommendation made was that examination schedules should not be so framed as to exclude the *possibility* of the use of a text-book, in 1938 still unpublished, in which the proofs should be based on the postulate of similarity instead of on Playfair's axiom.

The long discussion of pp. 35 to 40, considered further on p. 130 of this report, is full of interest for the teacher and the mathematical specialist, and full of hints for the writer of a text-book on these lines, but contains no suggestion that pupils who have learnt in Stage B proofs depending on Playfair's axiom should in Stage C change round and base their proofs on the principle of similarity.

Solid Geometry. The 1923 Report warmly advocated the inclusion of some solid geometry with the ordinary course of plane geometry. It was there laid down that ' there ought to be no formal separation between plane and solid geometry ', and some very brief suggestions were made as to how this separation should be avoided. These may be summarised as follows :

The technical terms should be learnt at first by examination of solids and there should be a course of modelling with paper before formal instruction is begun. When Stage B has been reached in plane geometry the methods of Stage A should be continued for solids. In Stage B models should be made to be the solution of definite geometrical problems (e.g. the construction of a skew pyramid to a given specification), and ' solid forms should be analysed by the drawing of plans and elevations ', and ' a good deal of the new matter should be taught in the form of exercises on the propositions of plane geometry '.

It must be admitted that these recommendations have so far had very little effect, probably because they were not sufficiently detailed.

The advocacy of solid geometry by the Mathematical Association has been successful in that it has led to the introduction of solid figures into examination papers on trigonometry—it is rare to find a trigonometry paper nowadays without at least one problem in three dimensions—and to the introduction of the solid figure into mensuration questions, but in geometry papers solid figures rarely appear except for calculations connected with Pythagoras' theorem.

This is not as it should be, and the present report goes into some detail with suggestions.

In the discussion of the work of Stage A, many places are indicated where solid figures should be considered and in particular it is advocated that the ability to sketch cubes, pyramids and other such figures should be acquired by systematic instruction and practice at this stage, rather than picked up haphazard later in connection with the mensuration and trigonometry problems mentioned above. In Stage B the 1923 Report recommends that ' the new matter should be taught in the form of exercises on the propositions of plane geometry '. To develop this suggestion a collection of typical questions dealing with solid figures is given in this report, arranged under the various groups of theorems, with the advice that such exercises may conveniently replace a similar number of plane riders and add variety and interest (see p. 85).

It is further advocated that the sketching of solid forms, if it has not been done in Stage A, should be done in Stage B.

Some notes are also given (see p. 40) on the teaching of ' plan and elevation '.

So far the suggestions are intended to be immediately practicable and suitable for the use of all teachers of geometry. It is true that it may be necessary to prune some of the less essential parts of the plane geometry course to make room for the new matter and some suggestions are made about the way in which this may be done (see p. 66), but in the main it is urged that the appeal to the solid will throw so much fresh light on the plane figure that understanding will be increased and that in the end little time will have been lost; time may even have been saved.

So far, it must be repeated, *the suggestions are intended for all teachers of geometry.*

For the few teachers, however, who are strongly attracted by the idea that the creatures of a three-dimensional world should not continue to dwell in Flatland, and who have a free hand in the matter, some much more far-reaching suggestions are made in Appendix 5.

There it is urged that the whole course of geometry should be based primarily on the study and analysis of solid figures. Such a course will be found to lead more directly from the elements to descriptive or projective geometry (cutting out a good deal of Euclidean matter) and its general adoption would involve revolution rather than evolution of our present syllabus.

The Committee, though far from advocating the adoption of revolutionary measures by all teachers, hopes that some enthusiasts will be able to experiment on these lines, and feels strongly that to do so would be to render good service to the community and to mathematics.

Intuition and Logic. Geometrical power depends on two factors, to one of which franker recognition needs to be given than is customary. That attention to reasoning is necessary is true enough, but traditionally such exaggerated importance has been attached to this factor that the other, which is certainly of not less importance, has been too much neglected.

In the days when Euclid reigned supreme this was natural; the subject to be studied was not 'Geometry' but 'Euclid's Elements of Geometry', habitually abbreviated to 'Euclid' pure and simple.

Now Euclid's great book is emphatically a logical, systematic treatise on the elements of the subject rather than a book on the subject itself; it is not concerned with the general body of geometrical facts known in his time, still less of course with the far wider field known to-day, but with the logical articulation of the simplest of those facts. So in days when for school purposes 'geometry' meant 'Euclid' it was not unnatural that examining bodies should allow 'Logic' as an alternative subject, for the very essence of Euclid is the logical examination of facts already well known, not a natural method of learning those facts in the first instance.

Even to-day there are fields of advanced geometrical study in which the logical aspect is predominant or even exclusive. In geometries of four and more dimensions 'intuition' is unavailable and such geometries are therefore predominantly if not exclusively logical. In 'Axiomatik' the only safe method is deliberately to exclude intuition, lest unrecognised assumptions should slip in unawares.

But in elementary school work we are faced with a different problem, viz. the first recognition of geometrical facts and the development of the power of geometrical perception. Until this has been to some extent acquired, attempts to reason on the facts are constantly hampered. The paradox may almost be ventured that in many cases until a fact is obvious, argument will be of little avail.

Our recognition of this is of course checked by the fact that under school conditions a boy will often say 'It is obvious' to cover ignorance: as teachers used to say 'Whenever I see the word "obvious" in a Euclid paper I know there is something wrong'. But it remains true about a very large proportion of geometrical propositions that unless they are ' obvious ' to the boy, apart from ' proof ', there is something very defective in his geometrical perception. As examples for this statement consider the following :

A straight line is determined by two points, a plane by three.

If a line falls across two parallels, four of the resulting angles are equal, likewise the other four ; and the sum of a pair taken one from each set is two right angles.

If a pair of parallels are drawn across another pair, the quadrilateral so constructed has its opposite sides equal and also its opposite angles.

The angles at the base of an isosceles triangle are equal.

Two sides of a triangle are together greater than the third.

Two circles intersect at most in two points.

If two circles touch one another, the join of their centres passes through the point of contact.

There is one and only one circle passing through any three points (with an exception).

The line drawn through the mid-point of one side of a triangle parallel to the base bisects the other side.

A line parallel to the base of a triangle cuts off a triangle of the same shape as the original.

The perpendicular from the centre of a circle to a chord bisects the chord.

The two tangents from a point to a circle are equal.

At any point in a plane there is one and only one line perpendicular to the plane. Perpendiculars to the same plane are parallel.

If a line is perpendicular to two non-parallel lines in a plane, it is perpendicular to all.

As to the truth of all these and a host of other propositions of which they are typical, a boy ought to feel fully *convinced* by mere intuition : he should say quite truthfully ' It is obvious ' or ' I see it '. As regards some of them he will need time to turn them over in his mind, and to draw figures—often several figures ; but conviction should be reached without formal proof and by his own activity.

In all these cases the purpose of proof is not discovery or realisation of the fact ; and great harm may be, and is in fact, done by misusing proof and stifling intuition.

As a typical illustration of this mischief may be cited a recent examination experience. Candidates were asked to prove that if through A and B, the points of intersection of two circles, parallels were drawn cutting the circles in PAQ and RBS, then $PQ=RS$. In roughly half of one examiner's scripts the lines were drawn parallel to the line of centres. In the remainder of the scripts the lines were correctly drawn so that PA was conspicuously unequal to

RB and AQ unequal to BS. That in the former group candidates should prove $PA = RB$ and $AQ = BS$ was natural enough, though of course the proofs were wrong. But the damning fact was that in the latter group also the great majority proved $PA = RB$ and $AQ = BS$ in the face of their own well-drawn figures showing this to be untrue.

After this it was a most refreshing experience to overhear a girl saying apropos of Fig. 1, ' I can prove the triangles equal, but they aren't ' !

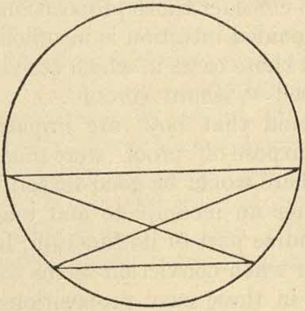

FIG. 1.

But the case of the candidates who drew bad, i.e. special case, figures and were thereby misled, illustrates another important point, viz. that beginners need much training and experience in drawing figures before their intuition becomes trustworthy, that is, before it avoids the danger of special cases. It is a familiar experience that if a class is told to draw a quadrilateral, many will draw squares, rectangles or parallelograms. At first this may seem to a teacher to be mere perversity or carelessness, but it is not necessarily so. The teacher is familiar with the significance of the figures, the beginner is not, and is not aware without experience, which he has not yet had, of the results of importing into a figure limitations and specialities which are not necessarily legitimate.

On the other hand beginners often draw figures which do not comply with the special conditions of a question and which therefore fail to help intuition ; equality of lines, parallelism and above all right angles are often wildly misrepresented. In general, freehand figures are preferable, but they must be so drawn as to represent or suggest the facts.

For both reasons much practice in figure drawing is necessary and it is very important that the question based on the figure should not always be answerable in the same way—contrast (as always) is invaluable in bringing out the essence of a situation. The question may be quite open : if you draw two pairs of parallels crossing one another what further properties will the figure have? There should be variations on this, e.g. if one pair is parallel, the other not, need either pair of sides be equal? Is it possible that one pair should be? How must the second pair be drawn to secure this?

The question may require the answer ‘ yes ’ or the answer ‘ no ’ : if the opposite sides of a hexagon are parallel are they necessarily equal? If two circles intersect do the common chord and the join of the centres bisect one another? Are the diagonals of a rectangle equal? Do they bisect the angles? etc.

Before going on to consider those propositions and groups of propositions in which unaided intuition is insufficient, we will consider the use of ‘ proof ’ in those cases in which conviction is attainable—and should be attained—without ‘ proof ’.

It is commonly said that boys are impatient of proving the obvious, and if the purpose of ‘ proof ’ were merely to prove that the proposition is true there would be good reason for such impatience. But that is altogether an incomplete and inadequate view of its function. It is of course part of its function, for intuition is by no means infallible even when conviction seems overwhelming—but it is only a part, and in these easy propositions not the important part.

Consider the propositions : a ball thrown into the air will come down again ; if a light is put to gunpowder there will be an explosion. Here experience, not indeed intuition, renders us fully convinced of the truth of the propositions, but the thoughtful mind does not rest there, but wants to know why these things are so.

Similarly in geometry, a certain construction produces unforeseen results ; the question presses—why does it do so? What is the connection between the construction and the consequences?

That this is a basic and permanent aspect of what we call ‘ proof ’ may perhaps be seen from more elaborate examples :

A narrow parallel-sided strip of paper carefully tied in a single knot flattened out without crumpling produces a regular pentagon. Why?

If the sides of two triangles taken in pairs intersect on a straight line, the joins of the vertices are concurrent. Why?

If P, Q, R are three points on one line, A, B, C three on another, the intersections of BR, CQ ; CP, AR ; AQ, BP are collinear. Why?

It is true that at least the two latter of these propositions and many others which will suggest themselves (intersection of medians of a triangle and the like) can never be *demonstrated by drawing*, for drawing is at best approximate and no experiment can demonstrate a truth of the Mathematics of Precision ; but in the cases quoted and many others the suggestion is so strong that the mind concludes that the truth is so. In other cases again the suggestion is weaker ;

there may be real doubt for instance whether lines that appear to intersect really and necessarily do so. In these cases, of course, ‘ proof ’ is necessary in the ordinary sense to settle a doubt, to confirm or to refute what may be a mere guess. But in all cases, even where conviction is complete, the mind wants ‘ proof ’ in the other sense of exposing the reasons for the ascertained result.

If the ‘ proofs ’ of obvious propositions are treated in this way, as explanations rather than mere confirmations, there is no reason whatever why they should be found dull. It does not require any very high level of intelligence to feel ‘ Yes, I see that it is so, but I should like to know why ’.

We pass now to the groups of propositions in which intuition unaided is not likely to arrive at the facts and where deductive proof as commonly envisaged is incontestably helpful. Does this mean that intuition ceases to function? By no means, but its rôle is modified.

(i) A proof may be perfectly correct and yet fail to produce real conviction because the learner cannot grasp it as a whole though he can follow it step by step. That proof is most effective which most closely approximates in impression to direct intuition.

(ii) In the search for a proof, still more in the search for new facts, logic alone is a very poor guide ; success in solving riders probably depends far more on intuition than on logic ; the path is blazed by the one, consolidated by the other.

Consider the group which deals with equivalent areas. The basic proposition, the equivalence of certain parallelograms, when first heard of, probably produces a shock of surprise, even incredulity ; the proposition as it stands is by no means ‘ obvious ’. But it is easily made so ; cut off a suitable piece from an end of one parallelogram and put it on to the other end and we have the second parallelogram. The formal proof is nothing but the elaboration of this simple observation.

In the subordinate propositions of the group the reasoning involved is of the slightest, e.g. the triangles are equal because they are halves of equal parallelograms. What difficulty there is lies wholly in recognising the various elements of the figure and their mutual relations ; which triangles are on the same base, which are between the same parallels, etc.

For this reason it is vitally important that boys should draw their own figures and do so step by step in order to get familiar with their build or growth, not first use the ready-made figures in the book nor merely copy them.

Direct recognition of the equivalence of figures of different shape is practically impossible : intuition cannot work unaided as it can to a great extent with congruent figures ; its function is to work on the figure in general, relying on the knowledge acquired by the propositions already established. Previously it was like a man set to work without tools other than his own limbs; now it is gathering equipment the use of which it learns by practice and experience; but intuition is still the workman, the living agent which sees, chooses, directs.

In another group, that connecting angles and arcs of circles, the case is closely similar, but intuition is not hampered by the difficulty of recognising by eye the equalities involved; equal areas cannot be recognised with any certainty, equal angles can.

To make the fundamental proposition 'the angle at the centre is double an angle at the circumference' almost obvious, all that is required is again a bit of construction: join the vertex of the angle at the circumference to the centre. This reveals two isosceles triangles and the consequent reasoning is child's play—provided the angles can be properly seen.

In the subordinate propositions and in riders the main difficulty, as all teachers know, lies in picking out the proper angles, seeing clearly which do stand on the same arc and so on; the reasoning used is of the simplest; the actual mistakes of reasoning commonly made in this section are of the type 'the angles are supplementary because they are supplements of the same angle'—which is itself due to not seeing clearly the relationships involved and relying on empty words.

In dealing with similar figures we get back to much the same state of things as we had in dealing with congruence; similar triangles are easily recognised if they are favourably situated, not so easily when they are awkwardly related to one another; but the search for them largely reduces to the search for equal angles, so that there is a fairly clear path to follow if once the goal is known or suspected. To know what to look for requires previous knowledge of the basic propositions and this knowledge, of course, needs to be exact; but use of the knowledge, selection of the useful details or point of attack, again depends on general insight, that is, on the power of rightly seeing the figure in its build and detail—on intuition, armed indeed with an increased number of tools, but still ultimately on intuition.

It is true that as progress is made, logic and intuition interpenetrate and help one another. For instance, to find a proof for an alleged fact it is often useful to start from the alleged fact and work backwards: this will be true if that is; that will be if this other is, and so on. Here argument may be said to take the lead, but intuition is still required to pick out the successive points to be considered.

It may be worth while to consider how one or two outstanding propositions and their proofs may have been originally discovered.

It seems probable that the fact of Pythagoras' proposition was discovered, so to say inductively, from observation of the fact that triangles with sides of certain lengths were right angled. If so we have an apparent fact approximately established by experiment, like for instance the intersection of perpendiculars in a triangle. And the fact cries aloud for proof—that is for proof in the double sense of making the supposed fact certain and of explaining why the fact is so.

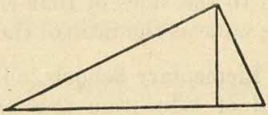

Fig. 2.

Heath concludes that the most probable supposition as to the discovery of the proof is that it started with the recognition of the similarity of the triangles, Fig. 2. This is almost mere intuition: the rest follows, to us, by easy algebra.

Take as another example the construction of the 36°, 72°, 72° triangle.

As given in Euclid IV 10 it is a veritable bolt from the blue, characteristic of his method of doing things without explaining why he does them. Why should one be interested in this particular triangle? And why should one begin by dividing a line in medial section?

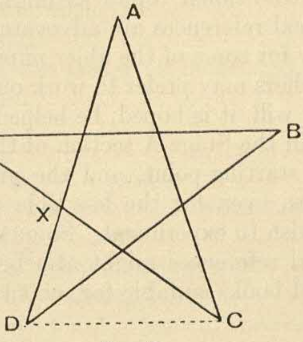

Fig. 3.

The pentagon and the dodecahedron were familiar figures and the urge to find a proper geometrical construction for the former is obvious. Playing with the drawing of a pentagon easily leads to the star pentagon (Fig. 3) and that to the triangle in question; how is the triangle to be constructed? Recognition of equiangular isosceles triangles ACD, CDX is easy and this gives $DX \, . \, DA = DC^2 = AX^2$ and the construction follows. This is a characteristic example of the interplay of intuition and reason ; intuition guiding but dependent on acquired knowledge to see the steps open to it.

Take as a last example Euclid VI 3 : the bisector of the vertical angle of a triangle divides the base in the ratio of the sides. I see two triangles and I notice that they have a common vertex and therefore I know that their areas are in the ratio of their bases. But I also notice that the same two triangles regarded as based on the sides of the original triangle have equal altitudes ; therefore their areas are proportional to the sides of that triangle ; therefore the ratio of the sides is the same as the ratio of the segments of the base.

Geometry in Senior Elementary Schools. In a booklet published in 1934 by the Board of Education (*Senior School Mathematics*, H.M.S.O. 1s.) it is stated that ' the comparative neglect of geometry is one of the chief defects in the teaching of Mathematics in Elementary Schools '. This neglect is attributed to the fact that to many people geometry is synonymous with Euclid, and it is proposed to remedy the defect by introducing a course on very similar lines to those suggested here for Stage A. The chief difference is that the course is an end in itself and not, as here, a preparation for later stages. It is recommended that the course should be based on surveying, machine drawing, the designing of patterns and other practical work. Solid geometry is introduced in connection with machine drawing, and it is suggested that teachers may decide to build their courses round such considerations as position, shape, size, locus, rather than such traditional topics as lines, angles, parallels and triangles. Historical references are advocated. Deductive work is recommended only for some of the abler pupils.

While some teachers may prefer to work out each his own course, the great majority will, it is hoped, be helped by the more detailed suggestions given in the Stage A section of this report. The use of solid figures as a starting-point, and the gradual introduction of deductive processes, even for the less able pupils, are ideas with which some may wish to experiment. Some of those who intend to introduce historical references might also be glad to know of the Association's list of books suitable for school libraries.

II STAGE A

Syllabus of Stage A. The content of this stage has twice been described by the Mathematical Association; in the 1923 Report on the *Teaching of Geometry in Schools*, and (in somewhat greater detail) in the report on the *Teaching of Mathematics in Preparatory Schools* (1924). The present report has only a little to add to what has been said there as regards the content. As however it goes into much greater detail about the various methods of teaching, it seems convenient for the sake of completeness first to give a syllabus in which the familiar matter is analysed somewhat differently.

The course consists of *Classroom work*, supplemented where circumstances permit by *Playground or Field work.*

Classroom Work. This may be analysed into the acquisition of

(i) certain ideas,
(ii) certain pieces of knowledge,
(iii) certain kinds of skill,
(iv) the rudiments of the power to make and understand geometrical argument.

(i) Ideas of

surface (as boundary of a solid);
line (as boundary of a surface), straight and curved lines;
point (as intersection of lines);
fixing position of a point in various ways, by the intersection of straight lines or curves;
the path or locus of a point, when only one condition is given;
figures, plane and solid;
the net or development of a solid figure;
angle as an element of shape, as a measure of amount of rotation and as difference of direction;
distance of point from line or plane;
parallels and the specification of direction;
congruence: the implications of the statement that two figures are congruent;

similarity: the implications of the statement that two figures are of the same shape or are similar;
symmetry, including that of the isosceles triangle;
area, volume, and the types of unit in which they are measured;

possibly also

the translation or sliding of a figure, e.g. the motion of a set-square and its edges as one edge slides on a ruler;
the rotation of a figure about an axis (i) perpendicular to its plane, (ii) in its plane;
tangency, the distinction between touching and cutting;
envelope, a curve as envelope of its tangents;
curvature, roundness of things as seen from the outside;
classification of the surfaces of the well-known solids according as straight lines can or cannot be drawn on them, and according as they can or cannot be developed.

(ii) Pieces of knowledge:

familiarity with the ordinary technical terms of geometry such as square, rectangle, right-angle, triangle, circle, isosceles triangle; with the meaning of vertical and horizontal, of line perpendicular to plane, of angle between plane and plane, and between line and plane; with the tests for straightness of line, flatness of surface;

and certain definite pieces of essentially geometrical knowledge to be employed in Stage B as a basis of deduction, viz.

the properties of the angles at a point,
the properties of angles made with parallel lines by a transversal,
the sets of conditions for the congruence of two triangles,
the sets of conditions for the similarity of two triangles, or at least the case of equal angles,
the angle-sum of a triangle (as this will be done by informal methods as well as by deduction it is included here), and perhaps of polygons.

[These last pieces of knowledge will have been acquired gradually during Stage A, perhaps unconsciously to some extent, but periodically what has been done will be reviewed and by the time that Stage B is beginning will be available as a basis for further progress.]

(iii) Kinds of skill:

the skill to use geometrical instruments,
viz. the ruler both as a straight edge and to measure lengths,
the set-square to make right-angles and perpendiculars,
the compasses to draw circles and to transfer distances (dividers),

the protractor to measure and make angles,
the set-square to draw parallels ;

to make nets of simple solids and models of them ;

to draw sketches of simple solids and to be able to see properties of the solids in the sketches, e.g. to realise that edges of a cuboid are perpendicular in the solid though they do not appear in the sketch to be so ;

to draw plans and diagrams from sufficient data (this includes the construction of triangles from any of the usual sets of data, the construction of various quadrilaterals, and so on) ;

to find areas by counting squares on squared paper and also by rule for rectangle, and perhaps for parallelogram, triangle ; for trapezium by dissection and later by rule ;

to find unknown lengths and angles by drawing to scale from sufficient data found, or supposed to be found, from measurement ;

to make the standard ruler and compasses constructions for bisection of lines and angles, for drawing perpendiculars and copying angles.

(iv) The power to make and understand pieces of geometrical argument will be acquired gradually as explained in the detailed account which follows.

Playground or Field Work. Outdoor or large-scale work is especially valuable for the interest it arouses. It also answers certain questions without direct teaching, e.g. is the size of an angle affected by the lengths of its arms? It is particularly useful to inculcate the idea of a locus. It imparts a sense of reality, which might otherwise be absent, to the classroom work which is done in connection with it.

For these reasons considerable stress was laid on outdoor work in the 1923 and 1924 Reports. The period that has elapsed since those dates has however considerably shaken the faith of many teachers in the value of this work and various criticisms of it have been made.

In the first place it is clear that outdoor work is much easier in the country than in the town ; also that it is much easier for the teacher of a small class than for the teacher of a large one.

Further it is difficult not to waste a good deal of time over it.

For these reasons the Committee suggests that most teachers will find it advisable only to use the outdoor work occasionally. Some of the outdoor exercises in geometry may conveniently be set as voluntary exercises to be done in their own time by any boys who are interested. So used they will prove stimulating to some without wasting the time of the others.

A few teachers may be favourably placed to make this a large part of their geometry teaching ; most teachers will be able to select some exercises which are specially suitable for their own circumstances. In any case constant reference should be made in the class to large-scale and outdoor applications of geometry.[1]

The Purpose and Content of Stage A. One of the great mistakes in the teaching of Mathematics, and one to which we are always liable, is that of presenting abstractions familiar to ourselves to minds unprepared for them.

The main purpose of Stage A is not so much to communicate knowledge as to help our pupils to make for themselves the abstractions with which geometry is concerned. They bring with them to school much experience relevant to our purpose, but this does not cover the ground and we have to give them work to do which affords the wider experience required.

What additional elements we have to supply depends on what they bring with them, and therefore, even disregarding their various degrees of ability, we cannot expect a uniform rate of progress or lay down in advance how many lessons will be required for a particular section of the work. Those for instance who have done any kind of constructional work bring with them a richer body of experience than those who have not.

While the work actually to be done by a form of beginners may vary there are certain fixed points of reference :

1. If it is to be the foundation for the usual course of metrical geometry, it must lead up to clear apprehension of the elementary propositions on which that course will be based, viz. the properties of angles at a point, the angle-properties of parallels and the conditions for the congruence of triangles.
2. It must lead to a clear sense of the difference between mere guess-work and the location of points and lines according to prescribed conditions.
3. There must be constant insistence on the necessity of seeing truth for oneself whether conviction rests on direct intuition or on argument.
4. Also on the precise use of technical terms and on boys expressing facts and arguments in language of their own. This leads up to but must not be replaced by learning ready-made statements from a book.
5. It involves the use of instruments, supplementary to but not replacing freehand drawing. The proper use of instruments, especially of set-square and protractor, needs careful teaching.

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[1] For detailed suggestions for outdoor work see the appendix to the M.A. report *The Teaching of Mathematics in Preparatory Schools.* This appendix is reprinted in the Preparatory Schools' Curriculum, published by the Headmasters' Conference.

The analysis and clarification of experience already acquired involves dealing, by questions not by dogmatic teaching, with such terms as solid, surface, line, point, straight, plane, curved, horizontal, north, distance, circle, square.

For other concepts, e.g. direction, angle, parallel, locus, fresh experience will generally have to be given.

The obvious fields in which the necessary experience can conveniently be obtained are :

the examination, construction and drawing of solids ;

'heights and distances questions', e.g. buried treasure problems, the interpretation and making of plans and maps.

The value of the latter is well established and current school books contain plenty of suitable questions making the path easy even for an inexperienced teacher. One caution perhaps is necessary : boys should be left to attempt the questions in the first instance unaided ; the reading of the question and its translation first into a freehand sketch to discover the meaning, and then into a measured drawing, are the most important parts of the exercise for the boys and should on no account be done for them.

As to the former, the construction of solids, there is less experience and less agreement. That some work of the sort should be done is perhaps generally accepted. Some think that very much more can be made of it and we add later in this report tentative suggestions which we hope some teachers will follow up, but we cannot recommend it with the same full confidence for general use ; there may prove to be technical difficulties in the way of its general adoption under ordinary classroom conditions.

However this may be, both fields alike contain plenty of material for gaining the experience necessary if the purposes which we have enumerated above are to be fulfilled.

But, we must repeat, experience alone is not sufficient ; it must be analysed and the results must be formulated in the basic propositions required for the subsequent deductive development.

As to the order in which the various topics should be taken up no general rule need or can be laid down. We are dealing with young children to whom activity of their own is properly more congenial than listening to a teacher or even than answering his questions ; therefore the sooner a class is set to doing something individually the better, whether it be making solids or doing heights and distances questions. It is very easy to bore a class, a fatal thing to do especially when starting a new subject, by too prolonged spells of explanation, question, or discussion. Therefore the more incidentally, we might almost say the more lightly, the less unfamiliar items occurring at the beginning of our list above are treated, the better it will be.

Though the newer topics, angle etc., will necessarily need more 'class teaching' this should be given in short spells; much time should be left for individual effort and experience, and the teacher should hold the floor as little as possible. Only when considerable experience has been gained by the class, i.e. towards the end of the course, should the major part of the time be taken for discussion, summing up and formulation.

The question of order on which there is most likely to be difference of opinion is the point at which angle shall be introduced. It is possible on the one hand to begin by giving 'hidden treasure' questions dependent on distance only, or on the other to deal with angle at once and so take all varieties of such questions together. It seems to be a matter for personal preference and either way is right provided it is made to lead up to the final conclusion, i.e. the conditions for congruence and the all important sense of the proper location of points.

Locus. We have laid stress on experience in the proper location of points: we must add that this needs analysing so that the concept of locus may be properly developed; and, as is the case with all concepts, this can only be done from a wide experience followed by reflection on that experience. A boy for instance fixes a point by the intersection of a line and a circle: with his eye on the final result he tends to lose sight of the separate intermediate steps of his thought and it is necessary that his mind should be recalled to these in isolation from one another. The first condition the point has to fulfil requires that it shall lie say on the line, but so far it may lie anywhere on the line. The second requires that it shall lie on the circle, but, so far as this condition alone is concerned, anywhere on that circle.

Frequent repetition of this sort of reflection on various types of material seems to be the natural way of forming and fixing the new and important concept. Variety of examples rather than elaborate discussion seems to be the prime necessity; and with such a treatment there is no difficulty, but great advantage, in introducing the idea quite early. We append a detailed discussion of this subject as of others to which we have referred.

Experience, Experiment and Argument. The main work in Stage A, we have urged, is the gaining of experience and reflection upon it. The latter almost necessarily involves some process of reasoning and by degrees leads on to a conscious use of argument. Experiences, at first disconnected, when reflected upon prove to be associated and even to be capable of reduction to a common source. There is no need either deliberately to seek occasions for argument, for they will occur naturally, or on the other hand to avoid them from the idea that argument is as yet premature and out of place.

But there is sometimes temptation to use argument where something else is properly required, especially in dealing with mistakes. A boy may assert for instance that the diagonal of a parallelogram necessarily bisects the angles; this is due either to his misinterpreting the word parallelogram or to his not using his eyes. The remedy is not argument or proof but demand for the figure he has in mind, and reconsideration of his answer.

Broadly speaking all questions as to the particular features of special types of quadrilaterals are of this kind; they require not only drawing, actual or imaginative, but also intuition. Nor are they matters of experiment: we do not ascertain that the diagonals of a parallelogram bisect one another by measurement, but by intuitive comparison of various figures. The proof, when we arrive at one, e.g. in Stage B, is an explanation of how the phenomenon arises from the build of the figure, much later perhaps, in C, developed into a complete analysis of the assumptions on which the proposition is based. It is necessary to draw a broad distinction between the legitimate use of measurement, e.g. in heights and distances questions for the approximate determination of the results obtained from data, themselves necessarily approximate, and its illegitimate use as establishing truths which for the present, i.e. until the axiomatic basis of geometry comes into question, must be regarded as absolute.

Ruler and Compasses Constructions. It is natural and desirable to include in the normal work of Stage A the elementary constructions by ruler and compasses, though in ordinary practice we may replace them by other means. These constructions are: bisection of an angle and of a straight line, the erection and the dropping of a perpendicular, the copying of a triangle and of an angle and the drawing of a parallel.

It should be noted that the erection of a perpendicular is a special case of the bisection of an angle, and that the bisection of a line does not necessarily involve the drawing of the perpendicular bisector—a useful but dangerous expression which often leads to confusion.

The necessity for good intersections is well illustrated by drawing them as they would appear under the microscope, ‘ lines ’ being represented by strips perhaps a centimetre wide, and the ‘ points ’ of intersection by parallelograms which grow to a calamitous length if the angle of intersection is too small.

An important factor in the accuracy of constructions is the ratio of the lengths of lines to their widths: therefore until somebody invents a boy who can sharpen a pencil, it is well to insist on long radii in such constructions as the bisection of an angle. A line is not accurately drawn [1] by joining two near points and then producing it many times its own length—a fact which is better appreciated when its statical equivalent has been tried, that is, that a long rod held by two strong boys at points near to one end can easily be displaced by one's little finger applied at the other. It is this defect which vitiates Euclid's construction for drawing a direct common tangent to two circles, if the difference between the two given radii is small.

Pattern Drawing. Some time may usefully be spent on copying and inventing patterns. This leads to a wider variety of constructions than that afforded by heights and distances and tempts to better craftsmanship and perhaps to greater care in the preservation of instruments, even to the desire for better ones. But it requires a much sharper look-out on the part of the teacher for guess-work and a stiff insistence on the proper determination of points and lines.

When the ground so far outlined has been covered it is easily possible and is probably desirable to proceed to the more formal work of Stage B, but there is still much to be done from time to time by Stage A methods, as will be seen from the list of topics given on pp. 13, 14. It is not necessary that these topics should be taken up at once. Areas and volumes will have presented themselves in connection with the study of solids ; other topics, e.g. symmetry, similarity, tangency, may well be left till they occur later. The important thing is that when they do occur they should not be treated from the outset in abstract mathematical fashion, as for example curvature is treated in text-books, but that there should be a reversion to what are in essence Stage A methods. As boys will then be less immature, less time will be needed for the necessary digression.

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[1] Or 'securely held', which emphasizes the connection with the statical experiment.

III DETAILS OF TEACHING IN STAGE A

Angles and Parallels. Probably everyone has a vague idea of angle as an element of shape and of the qualifying terms sharp and blunt, while right angles in particular are so familiar in experience that quite young boys easily recognise them. But for geometrical purposes something much more precise is required before angles are recognised as measurable quantities. It is on record that children when asked, after they had themselves pointed out right angles, to point out ' half a right angle ', answered ' don't be silly '. This illustrates very clearly the gap between the vague idea of shape and the precise idea of measurability.

The natural, perhaps the only way to the acquisition of the latter is by regarding an angle as an amount of turning and the best instrument to use in the first instance is just the human body. Standing stiffly upright a boy will easily execute a complete turn and then half and quarter turns. Thus the complete turn is the natural unit and the quarter turn is easily identified with the familiar right angle. Reference should of course be made to the drill commands, right turn, left turn, about turn, half right turn. Before any other instrument is introduced practice should be given in estimating angles between lines drawn on the blackboard as rough fractions of a complete turn or of a right angle.

The irrelevance of the lengths of the arms should be obvious if this mode of approach is used, but should be made explicit by a question.

The use of a circle for assisting estimation will probably suggest itself and further practice in estimation of angles drawn by boys themselves should follow.

Only then should the protractor be introduced and in the first instance in the semicircular form. Careful instruction is necessary as to putting it down correctly on the lines, especially with brass instruments where use of the wrong edge is common, and the habit of rough estimation should be steadily inculcated to check misreadings by confusion between the two readings on the protractor.

Parallel lines are familiar to boys from the outset, but their properties require analysis. The most familiar examples being afforded by the equally spaced parallels of an exercise book, or by the equidistant rails on a railway, this property of constancy of distance apart is naturally the one prominent in a boy's mind, and indeed it is the property enshrined in the name itself.

The negative property that parallels will not meet will of course be admitted ; but attention should be drawn to the fact that while any number of lines can be drawn through a point not meeting a given line, there is only one such line coplanar with the given line and we speak of *the* parallel through the point.

It is the angle properties which need most attention and naturally so, for angles are as yet rather new acquaintances.

It is essential that a boy should see for himself that when a transversal crosses parallel lines corresponding angles are equal and that all the angles involved are equal or supplementary. He will be helped to see this if the figure is contrasted with the corresponding figure with the lines not parallel.

No suggestion of doubt should be made : that belongs to a much later stage of work. Boys should be allowed to trust their own direct intuition that if the lines are parallel corresponding angles are equal and conversely that if the angles are equal the lines are parallel. As to the constancy of distance, however, questions do arise ; what do we mean by the distance between lines? Is there generally such a thing at all?

So far we take parallels as a familiar part of experience, merely enumerating their properties ; when for example in Stage B we begin to develop deduction and systematisation we shall have to choose one of the properties (viz. the angle property) as the basis and deduce the others from that. Not till then and indeed not till much later do we need any definition any more than we do of a straight line.

With angles and parallels are associated ideas of direction. Just as there are slight obscurities connected with the word ' straight ' (cf. ' leave the room straight ', ' go straight along the road ') which may need notice, so there are with the word ' direction ', which is used more vaguely in every day life than in geometry. People speak of going in the same direction if they are going towards the same point. This possible misunderstanding being cleared out of the way, if it occurs, the identity of parallels with lines in the same direction is accepted as a matter of course. Very useful exercises follow based on the ideas involved in such observations as : two men are walking north and each turns through 45° to his right ; their directions are again the same, viz. N.E.

This way of thinking is especially useful in dealing with the angle-sum of a triangle or polygon. From this point of view the most general form of the proposition, viz. that the sum of the exterior angles of any convex polygon is four right angles, is also the simplest. But of course the angle-sum of a triangle will be treated in other ways as well.

Reference should be made to large-scale phenomena on the earth; that north at New York is not parallel to north at London, that a triangle may have for example three right angles and so on. This is not to be done to throw doubt on what has been said or concluded, but to illustrate the difference between a sphere and a plane and to afford useful practice in geometrical thinking.

Congruence. The 1923 Report enunciates the Principle of Congruence thus: 'If instructions point to a single definite figure at one place, not only do they point to a single definite figure at another place, *but there are no internal differences between the two figures.*'

The possibility of such instructions rests on the assumption that there is meaning, as every one but a philosopher takes for granted, in the terms and statements characteristic of metric geometry: 'the line AB=the line XY', 'AB=3 inches', 'the angle ABC is a right angle', 'circle', 'square', etc.

Hence any proposition established by Euclid by the method of superposition can be established with precisely equal validity by noting whether, when it is re-stated as an order to construct a specific figure, the specification is such as points to a definite unambiguous result.

This latter process, the step by step construction of a figure, is much easier to a boy than the comparison of two figures 'given' or ready made. But if he is to extract from the process the propositions of congruence to which it points, it is essential that he should not merely execute the steps, but should reflect on what he is doing and note especially when freedom ends and consequences begin. It is here that the craft of the teacher is required. He must put such questions as direct attention to the essential point.

One effective way of doing this, one no doubt of many, is to say 'I want you all to draw a certain figure which I have in mind (say a quadrilateral). Here is a rough sketch as some guide, but the dimensions are not right; what dimensions shall I give you that you may draw it as I mean it to be?' Or the question may be inverted: 'If I tell you the base of my triangle is three inches, the left-hand angle is 50° and the right-hand side is 2½ inches, will that give an adequate specification?' Sometimes it will be necessary to have the drawing actually done; but it should often be sufficient to have it done in imagination merely. The essential thing is so to conduct the process that boys are not merely drawing figures to order, but are noting whether the specification is satisfactory—whether it is insufficient or redundant or leads to ambiguity or contradiction.

We say figures advisedly, for it is just as easy to take a variety as to confine attention to triangles and it is much more instructive. The difference is that as regards triangles the standard results must in the end be learnt as is the multiplication table.

Whether, as supposed so far, questions are stated in terms of abstract figures or are stated in more concrete form is a matter for the judgment of the teacher; methods will vary in detail in accordance with the task at the moment and the capacity of the class. The use of abstract figures has the advantage that it yields results very quickly; but this may be a danger when dealing with the slower moving classes. For those who feel the need for guidance in connecting this work with concrete situations the following suggestions may be useful.

A good problem would be as follows:

On my estate there are three trees, a beech, a willow and an oak (Fig. 4). I wish to draw a plan showing them. A lake wider than the length of my measure separates the beech and the willow. What measurements must I make in order to draw the plan? [All tricks with boats and string are barred.]

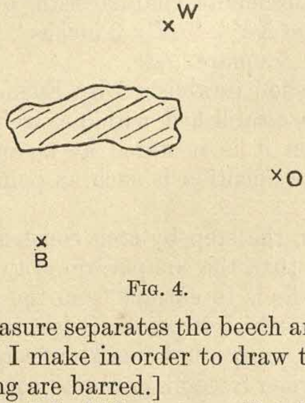


FIG. 4.

The measurements *BO*, *OW* are immediately suggested and discussion is automatically focussed on the angle at *O*. Boys can stand up in the classroom to represent the trees and the angle can be measured by using three pins and a plane table. The question of what will happen if the angle at *B* is given should be raised by the teacher, if it has not been raised by the class, in all but the weakest classes. With the problem in this concrete form it is not hard for the boy to draw a new figure and to see that there are two possible positions for *W*. This done the importance of the 'includedness' of the angle is driven home. (The measurements should be so arranged that the two positions for *W* will actually occur, it is too soon for any *discussion* of ambiguous cases. One point only is to be made, viz. that if the angle is included the triangle is determined.)

The next step is to put this result into a suitable form of words—‘ A triangle is fixed in shape and size if two of its sides and the included angle are given ’. To this the three side case is added by the use of some similar problem.

This work can be driven home further by questions such as this—‘ How many triangles of different shape can be drawn with the one side 3 ins., another 2 ins., and one angle 30° ’—to be answered by trial with freehand sketches.

The way is now clear for the following steps.

(i) The solution of the original problem when the lake is replaced by a river, as shown in Fig. 5.

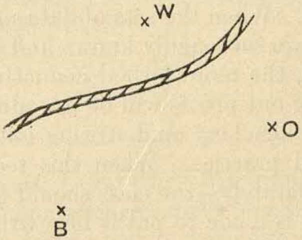


FIG. 5.

(ii) The statement for the case of one side and the angles adjacent to that side.

(iii) Investigation of possibilities when the positions of the angles relative to the sides are not stated.

(iv) The case of one side, one angle adjacent and one angle opposite to that side. The appropriate problem might be—given the length of the pole *AP* (Fig. 6) which is leaning towards the sun so that it makes an angle 120° with the shadow *AS*, the sun being at an elevation of 40° to the horizon, find the length of the shadow.

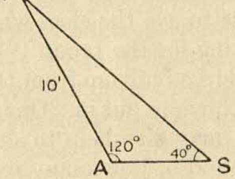


FIG. 6.

(v) The statement of this case and the case of one side and two adjacent angles in one form of words—‘ A triangle is fixed in shape and size if one of its sides and two of its angles and their positions are given ’.

(vi) Consideration of other figures than triangles.

This work will be done as part of the early work in drawing, and the class will probably not be ready yet for the use of the congruence argument. The natural elaboration is to give simple examples in the triangulation method of surveying.

When the use of congruence in argument is to be taught the preceding results will have to be re-stated as the usual conditions for congruence. There is no difficulty in seeing that conditions which determine the shape and size of a triangle will, if used more than once, yield triangles equal in all respects. This is really the beginning of the transition to Stage B.

It is important that much oral practice should be given in the use of the congruence argument before any attempt is made to produce a written proof. When the sets of data necessary for the congruence of triangles are thoroughly known and some facility in their use has been gained, the more formal deductive work and the accepted way of setting out proofs will be introduced. The technique of this process needs teaching, and writing out easy riders on congruence affords good practice. When this technique is taught it should be taught separately—the class should know what the proof is before an attempt is made to put it into writing. Otherwise the teacher and the boy will be ignorant whether the difficulty lies in appreciation of the proof or in the presentation of it.

It is also necessary to stress the limits of deduction ; the phrase ‘ therefore the triangles are congruent ’ is not a magic formula allowing one to say that therefore two lines are parallel, without any intermediate step.

Additional Hints for teaching Congruence.

(i) It is useful in the early stages to give a dimensioned sketch instead of the usual form, $AB=CD$, etc.

(ii) It is worth while to ask the class what they think might be proved equal before asking for the proof.

(iii) The figure should be built up from the data, equal lines and angles being marked as they are put in. One stroke for the first pair, two for the second, etc. may also help to show when the necessary three pairs have been found or how many remain to be found.

(iv) The first step towards the proof is to pick out two triangles containing the quantities to be proved equal. These triangles should then be written down, so that the proof begins : ‘ In the triangles . . . .’

(v) It is in this choice of triangles that the first difficulty crops up, and the early riders will inevitably include some that seem ‘ obvious ’ to the boy, often from considerations of symmetry. In these riders, which are more useful for oral than for written work, the necessary data usually stand out particularly clearly, and the difference between the 'obviousness' of intuition and the logical sequence of proof can be brought home.

(vi) The process can be made as real as possible by asking whether the triangles could be made to coincide by translation or if one would need to be turned over first.

(vii) The more difficult riders are those in which the required triangles overlap or in which the sides of one triangle form parts of the sides of the other. It is often helpful to dissect the figure and to draw the triangles separately or to outline them in different colours. The data will then show up more clearly.

(viii) It makes for clearness in the proof if the parts of one triangle are always written down on the same side of the equations, e.g. in the triangles ABC, PQR,

$$AB=PQ,$$
$$BC=QR,$$
$$\angle B=\angle Q.$$

(ix) It is recommended that when the triangles are stated to be congruent the case of congruence should be added as a reference in the way that Euclid refers to one of the congruence theorems. For this purpose the letters *SSS*, *SAS*, etc. should be enough. (See p. 55.)

Similarity. In scale drawing we are tacitly assuming the idea of similarity, and easy numerical work on similar figures will give the boy a clearer appreciation of what a scale drawing means. It is noteworthy that the study of congruence leads naturally to a consideration of similarity, and many teachers find it advantageous to make an informal study of similarity immediately after the work on congruence. It is, in any case, almost impossible to avoid considering the fact that the angles of a triangle determine its shape, but not its size, for so many boys are apt to assume a three angle case of congruence. Arithmetical examples in similarity can certainly go a long way towards overcoming this difficulty by connecting the knowledge of the three angles of a triangle with something positive which is not congruence.

The extent to which simple work on similarity can be done will depend on the progress that has been made in arithmetic and algebra. If sufficient has been done, this work will serve as a useful occasion for dealing with simple fractional equations. If however the arithmetical work is not strong enough for the teacher to proceed direct from a first treatment of congruence to a first treatment of similarity, care should be taken that, at some later date, similarity has its own Stage A. Similarity is one of the leading concepts of geometry, and the establishing of the concept is of greater importance than facility in the more formal rider work. Suggestions for early work on similarity are given on p. 90.

Geometrical Argument. Simple reasoning should occur as soon as possible ; but it should be, at first, almost exclusively oral. It should also be done without fuss ; if the argument is understood it should be left at that. Any attempt to make the boy fully conscious of the process that is being used will end in making a simple matter appear difficult. When the process has been used successfully for a long enough time for the boy to feel thoroughly confident of the conclusions reached by argument alone, then is the appropriate time to bring up the process itself for inspection. When this time is reached Stage B may be well established.

For the first arguments no questions of logical order should influence the teacher except those intrinsic to the particular argument under consideration. That is if the class will accept any premiss, no matter how wide in scope, any deductions from that premiss which the class understands are to be taken as valid without any examination of the premiss. The one thing necessary is a firm foundation of belief before any argument can usefully be attempted. The mathematician says ' If this is true, then that is true ' : the young boy has no use for the ' if '. Thus the following well-known method of showing that the sum of the angles of a triangle is two right angles is a perfectly suitable piece of work before congruence is reached.

Assume a diagonal of a rectangle divides it into two exactly equal triangles. Then in Fig. 7 it is easily proved that the sum of three angles of a right-angled triangle is two right angles and hence $\angle\alpha + \angle\beta =$ one right angle.

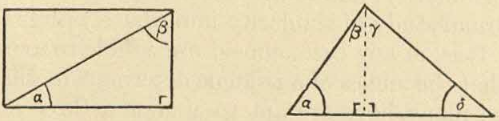


FIG. 7.

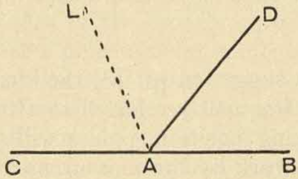
Any triangle can be divided into two right-angled triangles as shown, and hence $\angle\alpha + \angle\beta + \angle\gamma + \angle\delta =$ two right angles.

Of course all these steps would be done orally. There are tacit assumptions such as that any given right-angled triangle can be reached from some rectangle. There is only one remark to be made on this point—leave these assumptions alone at this stage.

Even though a boy knows that vertically opposite angles are equal from the ' scissors argument ', there is an advantage in showing him the more formal argument, perhaps not immediately following the scissors argument, but as a means of revising the fact when, a lesson later, he has forgotten it. Again, if one of the three fundamental angle properties of parallel lines has been established, the other two are generally best obtained by reasoning. It is quicker so, in addition to the advantage gained by familiarising the boy with geometrical reasoning.

It is when the meaning of ' angle ' is understood that small pieces of geometrical reasoning first begin to obtrude themselves to any considerable extent ; probably in the form of numerical calculations,

e.g.



$\angle BAD$ is 50°,
LA bisects $\angle CAD$;
find $\angle LAC$.

FIG. 8.

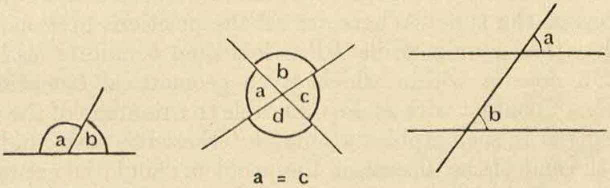
A useful first step for some beginners is to label each of the equal angles $x°$.

At this stage it is even possible to tackle the easiest exercises in general reasoning, such as the following :

' Prove that in Fig. 8, whatever the magnitude of $\angle BAD$ may be, the line AL is at right angles to the bisector of $\angle BAD$.'

The angle-sum of a triangle may here be introduced and will then give another fact to be used in numerical riders about angles.

From the first all facts that are to be learned should be learned in groups and should be exhibited in groups on the board. It is very helpful when exercises are to be done quickly to put a group of these facts on the board and give them temporary numbers, thus :



a + b = 2 rt. angles.
(1)

a = c
b = d
(2)

a = b
(3)

FIG. 9.

Then the class can be made to explore the figure they have in front of them by such questions as—' Where can you see figure (2) reproduced in your figure ? '. Again, when a boy is giving his reasons for the correctness of a proof which he is doing orally, he can give his references quickly by saying ' as in figure (3) ' etc. This is only recommended for early stages ; correct and concise wording for all facts should always be well learned. But the figure should be learned as well, or the words will be a rigmarole only.

Many general properties concerning angles can be obtained by exploring a figure having given one or more angles. Anything special that is noted can be tested by substituting x for one of the given angles and working round the figure in the usual way. This helps the algebra and it is surprising how much can be done in this way. A good class can easily establish the fundamental property of angles in a circle. As a closing remark on this part of the work we may say: always remember that the standard theorems and constructions are the best riders.

Loci. As has been suggested (p. 18) the idea of locus should be introduced early and frequently referred to afterwards.

In the work on fixing position, points will be determined with respect to some framework by two measurements or pieces of information, and it is good for the class from the beginning to consider what information is given by one of the measurements alone. When the position of treasure is being fixed by means of its distances from two trees, even the dullest boy will readily say what shape of trench he would dig to recover the treasure if only one distance were known. A scale plan of the trench drawn with a loop of thread and drawing-pin helps to make plain the use and advantage of compasses. When the plan has been drawn the question should be asked: 'What can be said about a point in the paper which is known to be three inches away from a fixed point of the paper?' and this can be followed up with the same question in three dimensions.

The corresponding procedure can be followed as each different way of fixing position is considered.

It is not necessary that all data should involve exact equalities. Questions of the type 'Where are all the positions of P such that AP is less than a units while BP is less than b units?' lead to the idea of a *domain* within which given geometrical conditions are satisfied. A contact with reality, suitable to this stage of the course, may be given to such problems (animals tethered to posts, and so on), but at the end of the discussion the problem should be expressed in formal geometrical terms.

The point of view of the last paragraph may be termed 'static'; the locus is regarded as a collection of points, rather than as the path of a moving point, the 'dynamic' point of view. It may be worth while, with a form of beginners, to introduce the study of loci with this latter point of view in the forefront, the paths being traced out for their own sake rather than for the fixing of individual points. It is worth while for the teacher to notice here the parallel lines of thought in connection with early work on graphs in algebra. The fixing of position of a point as the intersection of two loci corresponds to the graphic solution of simultaneous equations. The

Algebra Report states (p. 42) ‘ The main purpose of graphs is not the solution of equations but the exhibition of the behaviour of functions. The first necessity for a boy is to realise that with sufficient patience he can make a graph of any single-valued algebraic expression and that every such expression leads to an orderly curve. Before any graphic solution of linear equations is shown, the general idea of graphing functions of several types should be developed ’. In this respect the work on loci may afford useful support in the development of the idea of the orderly curve ; the boy is able to accept intuitively that a point moving under constraint, the nature of which is stated in geometrical terms, traces an orderly curve, more readily than he can accept the corresponding proposition stated algebraically. To call attention to the fact that the law of the curve may be stated in geometric or algebraic terms is in itself valuable as indicating the fundamental unity of mathematics ; of the three mathematical ‘ subjects ’ of lower school work, geometry is apt to be segregated.

In what follows the ‘ dynamic ’ point of view is more to the fore.

When the topic of parallels is reached, questions such as the following easily arise.

‘ What kind of path shall I describe if I walk along keeping always 3 feet away from a straight wall? ’

‘ What is the path of the centre of a bicycle wheel as the bicycle moves along on a straight piece of road? ’

These questions might be followed up with ‘ What is my path if I walk round a circular pond keeping always 3 feet from the edge? ’

‘ What is the locus of the centre of a ball which runs round inside or outside another? ’

‘ Sketch the path of the end of a string which is unwound (being kept taut) from a hexagonal or circular pencil after having been wrapped round it.’

All boys should have practical experience of drawing some loci other than straight lines or circles. An example or two early in the course will stimulate interest and, incidentally, will prevent the formation of the idea that all loci are straight lines or circles. For example, if a circle is drawn by means of a loop of thread passing round a drawing-pin and kept taut by the pencil point, as has been suggested above, it is a simple step to put a second drawing-pin within the loop, and to note that the path of the pencil point becomes an oval curve. It is quite possible to get from a young class the geometrical condition satisfied by points on the curve. One may well digress and suggest that this is the curve seen when a circle is looked at obliquely, or mention the orbits of the earth and planets round the sun.

One or other of the mechanical examples given below may also be used as an example of a locus which is neither straight line nor circle.

At a later stage, perhaps when it is desired to give exercises on loci by Stage A methods in conjunction with deductive work, it may be useful to plot a parabola and ellipse on squared paper from the focus-directrix property. Simple cases of Apollonius' Circle may also be tried (ratios of 5 : 3 or 2 : 1 are quite convenient).

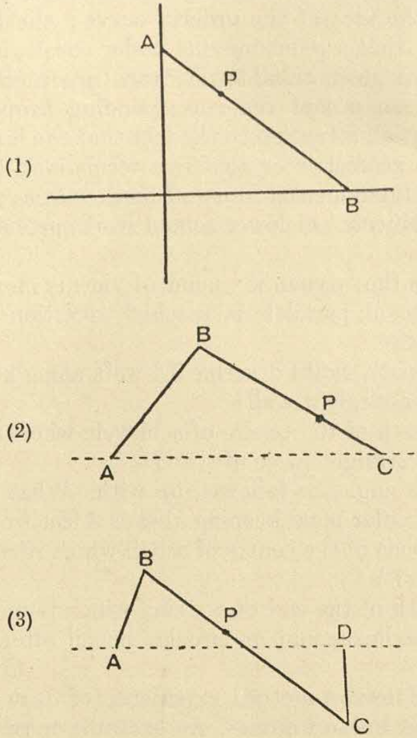


Fig. 10.

It is also interesting and useful to obtain some loci by mechanical means. Those which are suggested by Fig. 10 can be drawn with strips of metal and accessories from a meccano set, or with thin strips of wood, $\frac{3}{8}$ to $\frac{1}{2}$-inch wide, drawing-pins being used as fixed or movable hinges, and a small hole made in a strip for the pencil point.

1. The strip AB slides so that A, B move along fixed lines which may or may not be perpendicular. Trace the locus of a point P, fixed in AB.

The strip may be replaced by a 60° set-square, hypotenuse AB, and the locus of the right-angled corner may be drawn.

2. A is fixed, AB rotates about A, and C moves along fixed line AC. Draw the locus of a point P in BC.

Compare the cases $AB < BC$ and $AB > BC$.

3. A, D are fixed hinges, B and C movable. Draw the locus of a point P in BC.

If P is the mid-point of BC, and $AB = CD$, the locus is a symmetrical ' figure of eight ' which takes different shapes according to the distance AD.

Further shapes are obtained if $AB \neq CD$ or P is not the mid-point of BC.

It is worth while in these examples to make an attempt to describe in geometrical language the conditions which govern the motion of P.

One may also, with the more intelligent pupils at any rate, give some notion of the locus of a point which is given implicitly (e.g. as the intersection of two lines appearing in the figure). Apart from their value in extending slightly the conception of a locus, such questions often contain practical difficulties in drawing, which the abler pupils will take an interest in overcoming.

A good example of such a question is to trace the motion of the point of intersection of the rays from two searchlights revolving at given angular speeds. (The case in which one revolves at twice the speed of the second and in the opposite direction will be found to give a satisfactory curve, if a sufficiently large sheet of paper is used.) A harder example is the locus of the intersection of the lines joining the extremities of a revolving diameter of a circle to two fixed points on a fixed diameter. The result is a somewhat unexpected circle, the easiest case being that in which the two fixed points are on opposite sides of the circle. Either of these examples can easily be illustrated by meccano or similar apparatus.

Problems of this type can be made to serve as a useful introduction to the two-fold nature of the proof required for a locus property. A certain number of intersections having been found, an approximate idea of the locus can be obtained. The curve is now drawn freehand (or if it seems to be a circle, constructed by taking three of the points). Other points on the locus should then be chosen and tested to discover whether they also enjoy the geometrical property from which the original points were derived.

Envelopes. If time permits, some useful work may be done on envelopes and such work may very well prove to be the best form of introduction to tangency. Here, too, the first ideas should be ' static ', the envelope being considered simply as a shape which arises when a large number of positions of a certain straight line are drawn. The consideration of the mutual relation of the straight lines and the curve at once gives rise to the notion of tangency.

There are many easy examples which can be undertaken at this stage. A circle cut out in paper and folded so that the circumference always passes through a fixed point gives a clear ellipse, while a set-square sliding against a pin and having its right angle constrained to move along a fixed ruler or straight line gives a parabola. As in the case of locus questions an attempt should always be made to state the problem in abstract geometrical terms, but it is best to do this after the drawing has been done.

A ladder sliding between a wall and the ground is interesting as a practical illustration, but a better-looking drawing of the four-cusped hypo-cycloid is obtained by means of Fig. 11.

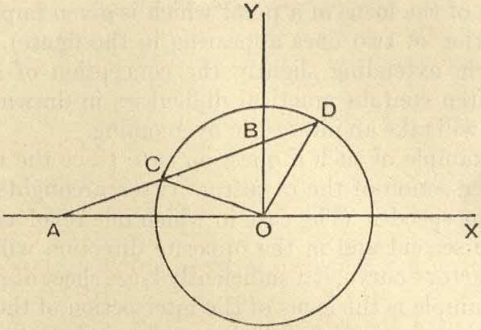


Fig. 11.

OY is perpendicular to AOX. CO, DO are radii of a circle, centre O, such that $\angle DOX = 3\angle COA$. The chord CD meets OX and OY at A and B respectively. It is easy, using angle properties, to show that $AC = CB = CO$, so that AB (the ladder) is of constant length. The drawing may be obtained by stepping off a convenient arc (such as that subtending an angle of 5° at the centre) round the circumference of the circle starting from the point where it meets OX, and then joining corresponding positions of C and D.

The class should then be invited to rephrase the question in terms of the motion of the points C and D, an exercise involving a certain scope for the imagination in the invention of suitable practical data.

Similar work may also be done on the simplest cases of the caustic curve. Here also a certain amount of preliminary work is advisable. In the case of a beam of parallel rays perpendicular to AO striking a circular mirror, it is easy to prove that $\angle AOC = 3\angle AOB$, and to complete the solution as in the previous example.

This gives a very good diagram if a sufficiently large circle is chosen, and incidentally enables the pupil to explain to himself the curve he can see in his tea-cup.

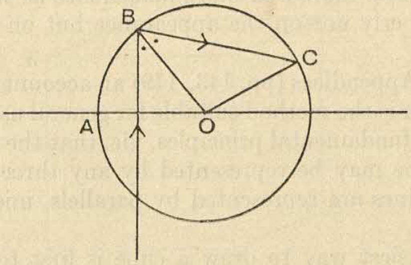


FIG. 12.

Solid Geometry. Some, but by no means all, of the advantages to be gained from the study and construction of solid figures were enumerated in the 1923 Report.

Acquisition of skill in the use of instruments ; the names and properties of geometrical figures ; the ordinary constructions for bisecting lines and drawing perpendiculars ; practice in analysing solid forms ; principles of scale drawing which underlie the theory of similar figures ; and (most important of all) the laying of a 'foundation of intuitive knowledge upon which a stable structure of theory may subsequently be built'.

Recognition of the value of later work of the sort is given on p. 24. 'Boys in Stage B should make models which are to be solutions of definite geometrical problems : e.g. the construction of a skew pyramid to a given specification. Later they should study the analysis of solid forms by drawing plans and elevations. . . .'

Perhaps the main contention in the following paragraphs is that much of the work envisaged in the sentence just quoted belongs quite properly to the early stages and affords in fact the natural introduction to formal work. As outlined in what follows it centres on the cube, but of course it admits of endless modifications.

Since much of the difficulty of solid geometry lies in the necessity of representing the facts in a two dimensional drawing and conversely of interpreting a drawing as their representation, it is of vital importance that from the outset practice in drawing and the reading of drawings should accompany construction.

First lessons may well be derived from the construction of a cube and practice in drawing the cube, and it will simplify subsequent discussion if something is said about this fundamental exercise.

To draw the net or development of a cube is a simple matter easily done for himself by almost any child ; and the less instruction or guidance given to him the better. But the drawing of one needs careful attention. Mere 'model drawing', i.e. drawing by eye with the model in view, is not adequate, even in this simple fundamental case, and becomes more and more inadequate as we proceed. The stress lies properly not on the appearance but on the build of the figure.

We give in Appendices (pp. 143, 149) an account of 'parallel projection', which is the method suitable for general use. Here we need only state the fundamental principles, viz. that the three concurrent edges of a cube may be represented by any three radiating lines; that parallel lines are represented by parallels, mid-points by mid-points.

Thus the easiest way to draw a cube is first to draw a square, then one receding edge and to complete the figure by drawing parallels. All the twelve edges should be drawn but the invisible ones should be dotted. For general use and for the first exercises in particular this will be the ordinary method of representation; but it is important later not to be slavishly tied to it. The fact is that any set of six parallelograms rightly arranged form a proper representation of a cube. This leads to some useful drawing exercises:

(i) *OA*, *OB*, *OC* (any three radiating lines) represent three concurrent edges of a cube; complete the figure.

(ii) *AB*, *BC*, *CD* represent three edges. . . . .

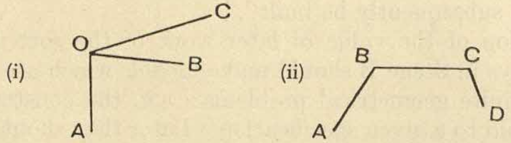


FIG. 13.

This fundamental figure of the cube itself gives rise to useful questions: e.g. do the two lines, this and that, actually intersect or does the one merely cross the other in the drawing? If they do not intersect, which lies behind the other? If the cube is drawn in

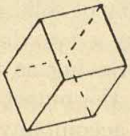


FIG. 14.

this particular fashion how are we looking at it; or where is it situated with regard to ourselves? But beyond these little questions, which recur with every new figure, there are many others which lead to fresh material. For instance: 'If you cut the cube in two by a saw cut beginning at *PS* and coming out at *BC* and remove the upper half, what do you get left? Make a separate drawing of it. Make its net.'

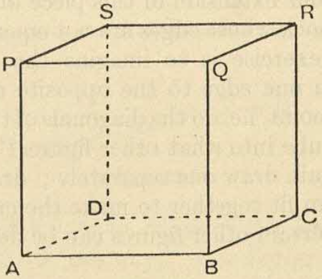


FIG. 15.

'Now take this new solid, which may be called a wedge [Are there any such things within your experience?], divide it by a cut through *SD* and *B*.'

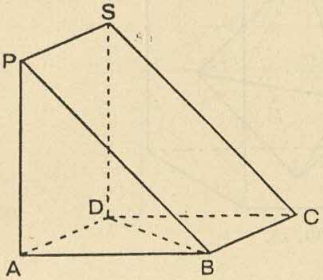


FIG. 16.

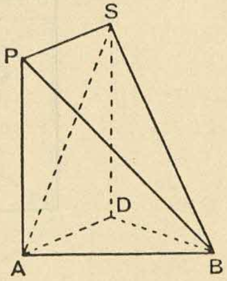


FIG. 17.

'Draw separately the two pieces which result; how would you describe them? [a square pyramid and a triangular one].

'Divide again the square pyramid by a cut through *S* and *AB*; draw separately the two figures which result; what are they?'

'We now have three triangular pyramids; are they in any sense identical in shape or size?'

This is an excellent *introductory* exercise on congruence; it involves identifying the six edges of each pyramid as being respectively three edges of the cube, two diagonals of the faces and one diagonal of the cube itself, and recognising that they are arranged in the same way except as to right hand and left hand in each of the three pyramids.

The nets should be drawn and the pyramids made (naturally, one only by each boy) and it may prove quite a puzzle to fit three of them together to form the wedge!

Notice that this involves experience of the fact that lengths not actually given can be obtained by drawing and measurement, a very necessary piece of experience.

There is an obvious extension of this piece of work, its repetition for a rectangular block whose edges are not equal.

Another useful exercise is to imagine the cube cut in all six possible ways from one edge to the opposite one [‘ Do these cuts meet in a common point, i.e. do the diagonals of the cube intersect? ’]. ‘ This divides the cube into what other figures?’ ‘ Square pyramids.’ ‘ How many?’ ‘ Again draw one separately ; draw its net and make it. Will six of them fit together to make the cube? ’

From the cube certain other figures can be derived by easy drawing. Thus :

‘ Draw a cube ; mark the centres of the faces ’ [‘ Has a square got a “ centre ”? How shall we find it in a properly drawn square? How in the parallelograms of our drawing? ’].‘ What figure do those six points suggest? ’ ‘ Join them in such a way as to make it.’ [Coloured chalks are invaluable here.]

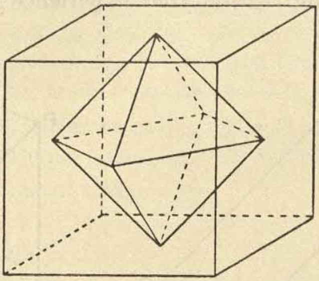


FIG. 18.

‘ How shall we describe this new figure? What are its faces? How many faces, edges and corners has it got? Can we find an independent way of drawing it? Draw its net and make it.’

Probably the drawing and construction of a regular square pyramid and also that of a regular triangular pyramid or tetrahedron will have preceded this last exercise. If so, we may take in a similar fashion the derivation of an octahedron from a tetrahedron by marking the mid-points of its edges, and of a tetrahedron from a cube by drawing diagonals of its faces, and finally, at least as a blackboard exercise, the derivation of an octahedron and two tetrahedra from the cube in a single figure. For boys with time on their hands, or at a later stage of work, the construction of the net of the two interlacing tetrahedra is a delightful and instructive piece of work.

As a piece of drawing the converse exercise is valuable : AB and CD are two opposite edges of a tetrahedron ; complete the cube from which it is cut.

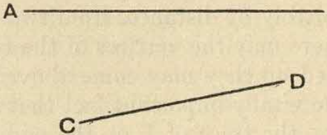


FIG. 19.

Yet another useful line is to consider the results of cutting off corners of the cube in various ways. It is better not to attempt too much at once, e.g. be content at first to ask for a drawing with say two corners only truncated—and those not too deeply (Fig. 20).

There are two cases of special interest : when the cuts pass through the mid-points of the edges, and when each of the original square faces becomes a regular octagon.

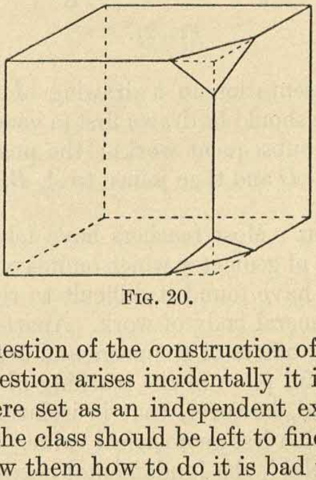


FIG. 20.

This raises the question of the construction of a regular octagon ; and because the question arises incidentally it is much more interesting than if it were set as an independent exercise. If possible (and it is possible) the class should be left to find a construction for themselves ; to show them how to do it is bad teaching. The nets of one or other of these truncated cubes (half regular, or Archimedian, solids) may well be drawn even if the figures are not actually constructed.

So far the figures constructed have been more or less regular ; these should be followed or accompanied by others which bring out more emphatically certain principles of construction. Even the construction of a regular block with unequal edges involves greater care in identifying the dimensions proper to each rectangle ; but a much bigger step is involved in the construction of an irregular pyramid.

'Make a pyramid whose base is a square (side 2 inches) and whose vertex is above a point other than the centre; but how shall I dictate to you the particular point I intend you to take? . . .'

Having settled this initial difficulty of the determination of a particular point (probably by distances from two sides) we meet the important one: Where may the vertices of the triangles be so that when the net is folded up they may come above O? This involves the recognition of the vitally important fact that as VAB is rotated round AB as a hinge, the trace of V on the paper is a line perpendicular to AB.

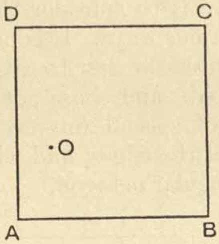


FIG. 21.

The correct representation in a drawing of such a pyramid is important: the base should be drawn first (a general principle which recurs frequently in subsequent work); the position of O properly settled; V put above O and then joined to A, B, C and D.

Plan and Elevation. Most teachers have felt uneasy about the adequacy of a course of geometry which omits consideration of 'plan and elevation', but have found it difficult to relate this branch of the subject to the general body of work. Apart from its utilitarian value in the drawing offices of engineers and architects, 'plan and elevation' has a useful place in the teaching of solid geometry; it should be emphasised, however, that in isolation from the rest of the subject it has much less geometrical value.

In 'thinking in solid', the fundamental necessity is the analysis of the solid in terms of *points* on its boundaries, each point being referred to each of three axes of coordinates. This mode of perception does not come easily to the beginner, who usually perceives a solid, vaguely, in terms of the shapes of its faces. That this is so is evident when the simple exercise on drawing a cube in parallel projection (p. 36) is extended to the case of a cuboid surmounted by a triangular prism (a dog-kennel—Fig. 22) when it is found that beginners often try to draw the end of the prism by using the base angles of the triangle, instead of determining the apex by means of its coordinates.

It is obvious that no two-dimensional drawing can represent with complete truth a solid form. In any method of drawing which gives a single complete picture of a solid, there is distortion of shape in one plane at least. In ' plan and elevation ', the shapes of figures in planes parallel to the three principal planes are preserved undistorted, at the expense of having to dissect the drawing into three parts. The preservation of the right angles between the axes has the additional merit of simplifying the actual construction of the drawing.

Generally, beginners find it easier to get an idea of the form of a solid from a drawing in parallel projection or in perspective than from a plan and elevation ; though the latter is easier to draw.

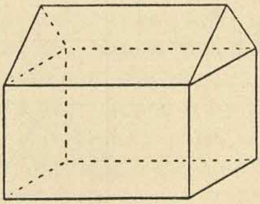


FIG. 22.

Work in ' plan and elevation ' should proceed concurrently with the sketching in solid geometry. Plans are within the everyday experience of the boy ; elevations are produced naturally by any small child drawing a picture, for instance a locomotive or a steamship. (The child's ' natural ' drawing of an aeroplane is usually a plan.) The production of end elevations is often quite interesting in itself. For exercises in reading plans and elevations, one may press into service advertisement literature such as that of housing concerns and some motor manufacturers. Having got the idea of the three views, the vital geometrical point is reached when the three views are mutually related by the projectors,[1] when it is realised that the plan and elevations taken together give a complete account of the solid. It is pointed out at the same time that, in making the drawing, the use of the projectors effects an economy in the number of measurements to be set out. Alternatively the economy motive may be introduced first, the importance of the projectors being brought in later. It is important to include as projectors the circular arcs connecting plan and end elevation. In either rough sketches or finished ruled drawings the correct drawing of projectors is to be regarded as equally important with the production of the three views comprising the whole.

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[1] The word projector is used here to indicate the dotted lines in the figure which show the connections between points of the plan and elevations, and includes the circular arcs.

A useful teaching aid consists of three boards to represent the horizontal and two vertical planes; fine knitting needles or wires may be used as actual projectors. When the three views have been traced on the boards, these are opened out flat, when the plan and elevations appear in their appropriate positions. The boards may be hinged together in two different ways, to give plan and elevations in the positions usual in English (Fig. 23*a*) and American (Fig. 23*b*) drawing office practice respectively. The diagrams show the particular usefulness of each convention.

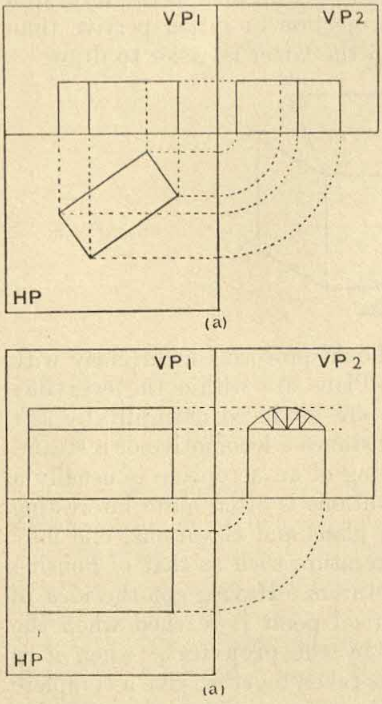

FIG. 23*a*.

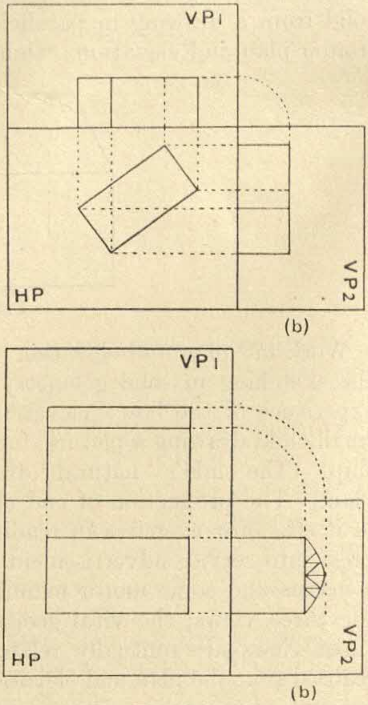

FIG. 23*b*.

For exercises in drawing plans and elevations, the usual collection of geometrical models should be supplemented by an assortment of machine parts and carpentry or cabinet joints and models. Finished accurate drawings from such models should usually be preceded by dimensioned freehand sketches (on squared paper). Drawings and verbal descriptions may also provide data for drawing plans and elevations, the exercises in the case of the projections mentioned combining the reading of these with the construction of the required drawing. The ideal method of testing capacity to read a drawing is the construction of the solid shown—the ordinary task of the craftsman. Failing this procedure, which can have but limited application in the mathematics course, the reading of plans and elevations will involve their conversion into other projections, or their verbal description. In general, exercises involving verbal description are probably the severest test.

Symmetry. The 1923 *Report on the Teaching of Geometry* contained, on pages 48 and 49, a section in which it was argued that symmetry should have a larger place in teaching than has been usual. The way in which this might be attained was indicated in general terms. It is unnecessary to repeat the arguments of the 1923 Report, but it may be helpful to supplement the suggestions on teaching method.

The first lessons on symmetry may arise quite early during informal work. The aim of such lessons should be to make clear what is implied in phrases containing the words ' symmetry ' or ' symmetrical '. An attractive introduction is by means of ' ink-devils '. A few drops of ink are sprinkled on paper, and the inked page is folded upon itself. The resulting bizarre pattern is examined and the simple and interesting facts arising from symmetry about an axis are realised. If two or more colours of ink are used the effect is enhanced. Another introduction is by means of the relation of object and image in a plane mirror. A strip of looking-glass is held at right angles to a blackboard on which is some word or diagram, and the appearances of the blackboard and the image are compared. If two mirrors can be clamped at right angles the quadrupling of a single design may be shown leading to the idea of symmetry about a point (or about an axis perpendicular to the plane of the design).

Among the earliest illustrations of symmetrical objects will, of course, be those from solid objects, a pair of gloves, a pair of semi-detached villas. The letters of the alphabet when examined for symmetry show the two simple types in the letters S and T, and it may be convenient to speak of ' point-symmetry ' and ' line-symmetry ' in terms of these letters. Both types of symmetry may be brought more closely to the pupil if he is allowed to fold paper, and to cut out patterns from folded paper. One may thus lead up to the constructions for bisecting angles, arcs, lines, and for drawing perpendiculars. The ideas of symmetry thus acquired may be useful at a far later stage. For instance to draw the bisector of the angles formed by two lines AB, CD, which do not meet on the paper available, one may argue :

The bisector will be an axis of symmetry. Draw two lines, one parallel to AB, the other parallel to CD and at the same distance from it.

These lines will meet at P, a point on the axis of symmetry. Similarly a second point may be determined.

Symmetry about an axis will be associated with the isosceles triangle, the rhombus, kite, isosceles trapezium, and the circle with a diameter. It will be necessary to emphasize those cases in which the axis of symmetry is not vertical or horizontal, for realisation of symmetry is then far more difficult. For instance the fact that the mid-points of the sides of a rhombus form a rectangle is less apparent when the axes of the figure are not horizontal and vertical.

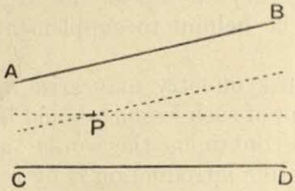

FIG. 24.

In all this early work the teacher should try to make explicit the geometrical consequences as to bisection of lines and perpendicularity implied by symmetry.

The recognition of symmetry is an important element in the development of intuition, but use of symmetry in formal proofs is particularly dangerous in early work.

The term 'congruent' may be used to describe the relation between two halves of a symmetrical figure, and it is well freely to use the terms 'symmetrical', 'congruent' and 'similar', as soon as the pupil can attach a meaning to these terms. In the past too much emphasis has probably been laid on one of these terms and too little on the other two.

In summing up the properties of the common quadrilaterals it is obviously convenient to refer them to any axis of symmetry the diagrams may possess or perhaps to a centre of symmetry. At the same time proofs should be stated in terms of congruence rather than in terms of symmetry, unless the teacher is prepared to construct a formal sequence of his own. This kind of work is probably commoner in Continental teaching than in our own. For an illustration we may refer to P. Chenevier, *Cours de Géométrie* (Hachette).

In later work the teacher should use symmetry freely whenever his natural mathematical instinct is to do so. For instance in plotting circles of Apollonius, using $AP=kPB$, one may point out that $AP=2PB$ and $2AP=PB$ give circles symmetrical with respect to the perpendicular bisector of AB. In allied subjects, e.g. graphs, one may use symmetry to sum up the relation of $y=ax$ and $y=-ax$

or the shapes of $y=x^2$ and $y=x^3$ or the relation between a function, e.g. 10^x or $\sin x$, and its inverse. For instance it is interesting to point out how the gradients of $\log x$ and e^x may be connected by considerations of symmetry.

The Use of Squared Paper. 1. The ruled lines on squared paper may replace the ruler as a measuring instrument in experimental work and scale drawing.

Thus to draw a triangle ABC with sides 2·7, 3·1, 2·4 inches: BC is drawn along a ruled line 3·1″ in length. With the compass point at B and legs opened along BC to a radius of 2·7″ an arc is drawn and similarly for the 2·4″ side.

In scale drawing the scale and the measurements represented can be marked in along a line.

2. In experimental work on proportion and similar triangles it may be used in the same way. This work follows work on using scales for such questions as heights and distance problems. It leads up to a discussion of the proportionality of sides of equiangular triangles.

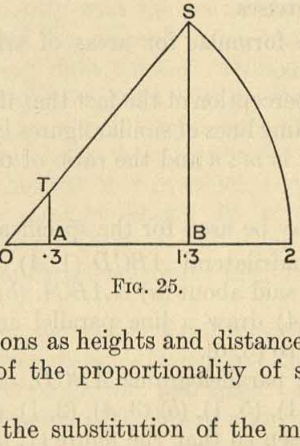

FIG. 25.

In all such work the substitution of the measurements on the squared paper for the boxwood ruler means an economy of time and some gain in accuracy.

3. It enables a pupil to draw a 'diagonal scale' with comparatively little labour.

4. It can be used to obtain such lengths as $\frac{6}{13}$ of an inch. In Fig. 25, OA is ·3″, OB 1·3″; a circle centre O radius 2″ meets BS, perpendicular to OB, at S; OS meets AT, perpendicular to OA, at T.

Then $\dfrac{OT}{OS}=\dfrac{3}{13}$, $\therefore\ OT=\dfrac{6}{13}$ inch.

5. Squared paper is probably the best vehicle for conveying the idea of what area is and how it is measured. In the fundamental case of the area of a rectangle, squared paper gives a visual demon-

stration of the rule, referring the pupil to the general principle, not always clearly recognised, that a quantity is measured by the number of standard units that can be fitted into it.

6. The use of squared paper is almost indispensable[1] in experimental work on area, such as:

(i) to discover the ratio of areas of circles in terms of the ratio of their radii and incidentally,
(ii) to find an approximate value of π;
(iii) to find the area of—say—Arizona, by tracing the map on squared paper, counting the squares and using the scale.

7. There are many boys who must see with the eye before seeing with the mind, who must acquire a conviction of a geometrical truth through the senses; there are some to whom purely deductive methods convey no concrete significance; and there are boys who have to be taught how to count squares.

All such boys will profit by a fuller course of squared paper work. It might include exercises:

(i) verifying the formulae for areas of triangles, trapezia and parallelograms.
(ii) leading to a perception of the fact that if the ratio of one pair of corresponding lines of similar figures is $m : n$, then the ratio of every pair is $m : n$ and the ratio of corresponding areas is $m^2 : n^2$.

8. Coordinates may be used for the specification of figures thus

(i) Draw the quadrilateral $ABCD$ (1, 4), (3, 1), (5, 4), (3, 5). What can be said about (*a*) $\triangle ABC$? (*b*) $\triangle$'s ABD, BCD?
(ii) Through (2, 4) draw a line parallel and equal to the line joining (1, 3) to (6, 0).
(iii) Complete the parallelograms $ABCD$, of which A, B, C are (*a*) (3, 4), (2, 1), (5, 1), (*b*) (3, 4), (2, 1), (5, 2).
(iv) Can y be chosen so that the figure $(1, y)$, (3, 1), $(5, y)$, (3, 5) may be a square?
(v) Find the area of the quadrilateral (2, 10), (5, 4), (10, 8), (8, 15).
(vi) Draw a pentagon with coordinates of 5 points given. Reduce it to an equivalent triangle.

9. Pictures of solid figures can be drawn by joining points specified by coordinates.

Thus the points (0, 3), (7, 2), (8, 5), (1, 6) will give a medial plane and (4, 0), (4, 8) the vertices of an octahedron.

In such work, where a figure is symmetrical, an acquaintance with negative coordinates (or some notation to serve the same purpose) will enable a specification to be made in which the symmetry will appear in the numerical value of the coordinates. Thus the above octahedron would be given as

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[1] **Except for the alternative method by weighing.**

$$(-4, -1), (3, -2), (4, 1), (-3, 2) \text{ and } (0, -4), (0, 4).$$

Minor Notes. *Fixing Position.* In the very early stage of plane geometry there is a satisfactory way of linking the lessons together.

When making maps and scale-drawings the prime necessity is to be able to locate a point. This *fixing of the position of a point in a plane* is the needed link. It involves ideas of loci and leads up to ideas of congruence.

Many teachers think that it is in connection with this and not in drawing the nets of solids that the ruler, compasses and set-square should be introduced.

The various methods by which a point is fixed can be taken in order and fit conveniently with the use of the various geometrical instruments. We may for example begin with the ruler and the set-square (for drawing right angles). Thus a framework or plan can be provided, on which a point is to be located. To recover a buried treasure provides a picturesque motive, and if this treasure is stated to lie where the path is intersected by the line joining two trees, we have the first way in which a point is fixed, by the intersection of two straight lines. In connection with these buried treasure problems there is real value in a little outdoor work in which a coin, thrust edgeways into the ground, has to be found.

The circle may now be used in fixing position. The ‘ treasure ’ will be buried at a given distance from a tree and in a line with two others or at given distances from two trees. Geometrically this is equivalent to fixing a point as the intersection of a circle with a line or with another circle.

The idea of fixing position is still to the fore while angle is being taught, primarily the fixing of the vertex of a triangle given the base and the angles at its ends, or fixing a point given what are later on called its polar coordinates.

Protractor. The protractor has two forms, semicircular and rect angular. There is no doubt that the former is the easier, and should be used at first.

The latter, however, besides being useful as a ruler enables an angle to be drawn without ‘ making a dot ’ first—though the method of doing so is by no means a natural one and needs to be taught.

Fig. 26 explains the method of making at O an angle of 70° with OA by ruling along OP.

Boys should be taught to make a mental estimate of the size of an angle before measuring it, and to visualise an angle of a given size before drawing it with a protractor. If this is done misuse of the protractor may be largely avoided.

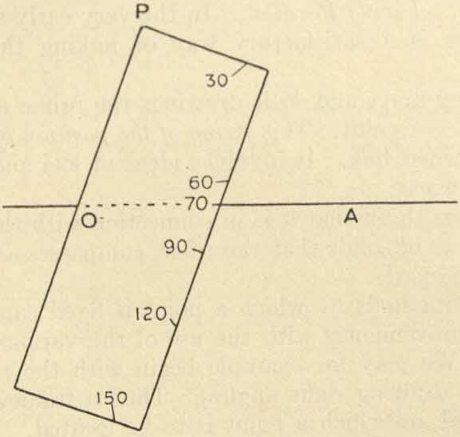


FIG. 26.

Note that the properties of angles at a point are implicit in the protractor and the double marking round it; the connection between straightness and supplementary angles should be brought out at this stage.

IV STAGE B

The Relation between the Stages. Stage B forms the main part of the geometrical work in schools. There is no definite break from Stage A work, but rather a change of emphasis which takes place gradually from six months to a year after geometry has been started. This assumes that Stage A work has been along the lines suggested in this report. Stage B gradually gives place to Stage C work with most pupils during the year before the school certificate examination is taken.

Many teachers have kept the stages of geometry so distinct that they seem to envisage some such arrangement as this :

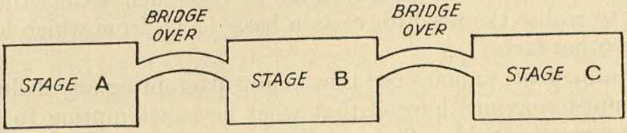


Fig. 27.

But the human mind does not grow in an orderly fashion such as that suggested above, nor is it always possible to be off with the old love before you are on with the new. The arrangement of Stages which we recommend is more properly represented like this :

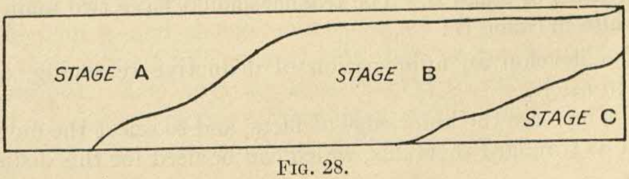


Fig. 28.

Stage A will continue to some extent (though perhaps more spasmodically than shown above) all through the course, while Stage C will begin gradually and will certainly not supplant Stage B entirely.

The change from Stage A to Stage B lies chiefly in the increased emphasis on deductive reasoning. As we have pointed out on pp. 5-12 training in intuition and in reasoning go together throughout the geometry course, and if the latter is now emphasised the former must not be neglected.

At the outset teachers are faced with a difficulty. ‘Obvious’ results are easy to prove, but to many do not seem worth proving. The less obvious results illustrate better the value of the method of deductive reasoning, but the proofs are less easy, and so not suitable to beginners. Consider as an example of the first type the proof of the isosceles triangle theorem by bisection of the vertical angle. The child sees that the angles opposite the equal sides must be equal, and does not need to prove it. But, on the other hand, he finds this a delightfully easy example on congruent triangles, and appreciates that here he obtains one fact from another fact which he knew. This is the essence of deduction, and boys will accept the process as a new method, or game, and be prepared to see what other facts, which they already know to be true, can be deduced or proved by the use of congruent triangles. A certain amount of work of this kind, not too much, will prepare them for the next step of proving less obvious and non-symmetrical properties. The line of attack might now be, ‘Here is something which looks reasonable enough, but I cannot be sure that it is true; can I use this new method to deduce it from facts I already know?’ By such means the boy begins to realise the need for certain basic facts, from which he can deduce other facts.

It is important to point out that before attempting any deduction a boy must convince himself that what he is attempting to prove is at any rate probable. This needs a reasonably well-drawn figure, or in some cases several figures.

Once the method of deduction is understood, boys should use it to prove facts which are better appreciated by this means than by any other.

Main Aims of Stage B. The teacher should have two main aims before him in Stage B:

(i) To develop an appreciation of deductive reasoning, and a power to use it.

(ii) To increase the knowledge of facts, and to select the more important as standard theorems, which can be used for this deductive process.

In order to secure these objects the following methods are adopted:

(*a*) Learning the results, and in some cases the proofs, of standard theorems.

(*b*) Using these results for calculations, and for the solution of riders, by the deductive process.

(*c*) Using the results of the theorems for constructions involving accurate deductive reasoning.
(*d*) Sketching figures for two and three dimensions sufficiently accurately to help intuition.

These methods are not to be taken in any particular order, but are essentially to be worked together. It must not be assumed that the proofs of theorems come first; insistence on learning the proofs first is without doubt the greatest mistake which the inexperienced make in teaching the subject. In particular the method of setting a completely new theorem for preparation for a later lesson is a bad one; in most cases it is better to set the preparation of the proofs of a group of theorems which have previously been discussed.

In the following comments on the four methods given above, it is convenient to consider them in the order *b*, *c*, *d*, *a*.

(*b*) *Riders.* The solution of riders is, in the opinion of many teachers, the most important of the four methods. Numerical calculations are often useful to lead up to theorems, and to drive home the knowledge of them, especially for the weaker boys. The solution of riders is the chief means of developing geometrical power, i.e. both the power of geometrical intuition and that of deductive reasoning. Every means should be used to encourage boys to solve simple riders, and also from known properties to find for themselves other properties, and justify them by reasoning. Ample use of spare paper should be allowed for drawing and marking figures. For further remarks on riders and the method of helping boys to do them, see p. 68.

(*c*) *Constructions.* The drawing of a figure whose construction needs the use of some theoretical fact is often the best way of implanting this fact. Thus the construction of accurate figures is of considerable value to everyone, and particularly to those who have little ability for theoretical geometry. Such boys often like drawing beautiful figures, and should be encouraged to draw them. This arouses their interest and so may help them to tackle other parts of the subject. Also boys are thus enabled to express their geometrical reasoning in figures rather than in words, and they may find this easier as a first stage. Constructions are considered in detail on p. 73.

(*d*) *Freehand Sketching.* This is most important and is apt to be overlooked by teachers. Boys should be encouraged to draw figures freehand as much as possible. Care should be taken to see that the figure is as general as possible, and is not a special case. This is essential if a boy is to spot facts which are likely to be true. Three-dimensional figures are more difficult and hence need more practice, and boys need to be taught how to draw them, but the drawing has a utilitarian value and demands imagination, and so is doubly valuable. Practical suggestions have been given on p. 35.

(*a*) *Standard Theorems.* To learn the results of theorems is essential since these provide the basis for deduction. At first it may be helpful to learn the particular enunciation of theorems by reference to figures rather than in words. But it is important that the boy should also learn to formulate general enunciations in good English. To do so develops his knowledge of geometrical language, and gives him practice in composition.

The proofs of some theorems need to be learned, inasmuch as there is a construction, or a part of the proof, which does not suggest itself automatically to the boy. This learning may have considerable value, especially for the weaker boy. When he has mastered the proof of a theorem and can reproduce it, he has a model of the deductive process. If he finds the solution of riders difficult, he will gain confidence from theorems, for he will feel that there is something that he does know. But at the same time one must remember that the learning of theorems is of less educational value than the solution of riders.

Of the time given to geometry, about a third may well be given to constructions, and of the rest at least half should be given to riders, if among riders are included numerical examples and proofs of propositions which will *subsequently* be regarded and learnt as standard theorems, but which, at the time the boy does them, are being regarded as deductions from previous theorems. For example, among riders on congruent triangles all the properties of the parallelogram will probably be worked out.

Just as it is a help in solving a rider to know from what theorem it is to be deduced, so, in following the proof of a theorem, it is essential that a boy should realise from what previous fact this theorem can be deduced. This knowledge is gained more easily if, in Stage B, a boy thinks of theorems in *groups*, each group having a fundamental theorem from which the other theorems of the group are deduced. Care should be taken that two theorems are not deduced from each other.

These groups will be considered in detail, but first it is convenient to divide theorems roughly into three classes. This mode of division may well be explained to the boys.

Classes of Theorems. (α) There is the large class in which the deductive process helps to introduce the boy to new facts, such as the angle properties of the circle. The proof is here in fact the best way of convincing the boy of the truth of the theorem. These theorems are sufficiently striking to merit attention by the boy, and he should work through them carefully, and be able to reproduce the proofs.

(β) There is a class of theorems whose results become apparent more by intuition or by symmetry than by proof. These theorems boys should be able to prove as easy examples of the deductive process, but to lay stress on such proofs may seem to them both unnecessary and irksome.

(γ) The third class includes those theorems whose results are important but whose proofs are difficult so that boys should not be expected to reproduce them at this stage. Such theorems are those on congruence and parallels, the theorem that a tangent is perpendicular to the radius, and perhaps the similar triangle theorems, and those converse theorems which need a proof by *reductio ad absurdum* (see p. 95). The number of such theorems whose results can be assumed will vary according to the calibre of the boys with whom the teacher is dealing ; but, in the case of the weaker boys, a teacher should without hesitation place a theorem in this class, at any rate for a first reading, if the application of the results is appreciably easier than its proof. Reproduction of these proofs follows more easily in the beginning of Stage C work.

In addition to the usual standard theorems included in these three classes there is the quite indefinite class of the theorems which are certain to be included in a geometry course if it goes far enough, whether as standard theorem, or standard rider, or property of a frequently occurring figure, e.g. the common chord of two circles is bisected at right angles by their line of centres ; the mid-points of the sides of any quadrilateral form a parallelogram ; the altitudes bisect the angles of the pedal triangle ; if p and d are the sides of a regular pentagon and decagon in a circle of radius r then $p^2=d^2+r^2$.

Of such theorems it may be said that by doing them as riders, or by blackboard discussion, the pupil gets to know the gist of the proof, and is able, if given time to think it out, to cast the proof into the usual formal arrangement.

Of these additional results, and perhaps of the theorems of group (γ), it may be said that the ideal to be aimed at is that the boy should be set to learn (or prepare) few proofs, but should get to know many. For further remarks on standard theorems, see p. 80.

Freedom of Order. In the following pages a fairly full and detailed syllabus is given, but it is not suggested that any attempt should be made to adhere to this rigidly. It is arranged in groups of theorems, and there is much freedom in the order in which the different groups can be taken. Emphasis should be laid rather on the grouping than on the links between the groups, the latter being the beginning of Stage C work. It may be said that after the congruence and parallelogram groups there are three main groups, the area group, the angle properties of the circle group, and the similar triangle group. Each has a claim to be taken first. Similar triangles follow easily and naturally after congruent triangles, though the number of riders is limited if the angle properties of the circle are not known. The latter provide the most delightful example of a group of theorems, and every teacher is anxious to have the joy of presenting them to a class. The area group is practical, and fits in with mensuration work in arithmetic. Hence, in the following statement, the importance lies in the grouping, and not in the order of the groups. As a guide to the teacher one of the letters α, β, γ is attached to each theorem quoted to suggest the class to which it belongs. Opinions may differ considerably as to whether these classifications are correct. In addition the fundamental theorem of each group is specified where possible.

V THE GROUPS OF THEOREMS

The Angle-sum Group. The fundamental theorem is that the angle-sum of a triangle is two right angles (α), proved from Fig. 29 (*a*) or Fig. 29 (*b*), and that the external angle is the sum of the two interior and opposite angles. It should be pointed out that these are two closely allied results. They should be confirmed in various

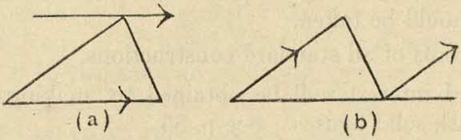


FIG. 29.

other ways such as measurement, rotation and folding, but the geometric proof has here the advantage of being easier and more convincing than any other way of obtaining the result, and the proof is especially suitable as a first model of deductive reasoning. Extensions to the polygon provide good examples for calculation and for riders, but opinions differ as to how far the results of formal theorems on the polygon should be learned. Knowledge of the isosceles triangle theorem (β) greatly increases the scope of riders and calculations.

The Congruence Group. A convenient notation by which to refer to the theorems in this group is that of using *S* for side and *A* for angle, so that *SAS* means side, angle, side in that order, i.e. two sides and the included angle. Hence the first three theorems are the *SAS, AAS, SSS* cases of congruence (not *ASS* or *SSA* as the pupil will readily appreciate). The fourth theorem may usefully be thought of as two sides and a right angle, in contrast to two sides and the included angle. To emphasise the fact that the angle is a right angle, it may be convenient to state the right angle first, and since the other two sides are the hypotenuse and one other side, an appropriate notation is *RHS* (or *RSH* or *RSS*). Any consideration of proofs may be postponed till Stage C. As regards the ambiguous case, boys should realise in Stage B from construction that, given two sides and a non-included angle, there are only two possibilities, and hence that if two such triangles each contain an obtuse angle they are congruent. Fuller consideration will naturally follow in Stage C*a*, when trigonometrical notation can be freely used.

In selecting riders it is important to include cases which are not obvious by symmetry, so that the boy may more fully appreciate the value of a deductive proof.

In addition to miscellaneous riders there are several standard types indicated :

(i) The properties of a parallelogram, and the conditions that a quadrilateral should be a parallelogram.

(ii) The properties of other special quadrilaterals, including the rhombus, kite and trapezium.

(iii) Anticipations of parts of subsequent theorems ; the mid-point and intercept theorems, equivalent parallelograms theorem, Pythagoras' theorem. Only the parts of these theorems dealing with congruence should be taken.

(iv) The proofs of all standard constructions.

Variety and interest will be obtained by making some of the riders deal with solid figures. See p. 85.

The Parallelogram Group. The theorems in this group, of which there are two—the properties of the parallelogram, and the conditions that a quadrilateral should be a parallelogram—each consisting of four or five parts, are such that all the parts can be and should be proved as riders on the congruence group (except for one belonging to the angle-sum group). These theorems belong to the α class.

The main difficulty for the boy is to realise the importance of working in one case *from* and in the other case *to* the definition, and of remembering the definition of parallelogram apart from its properties. This is almost the only definition in elementary geometry to which much attention need be paid. The fact that appreciation of a definition comes at a more mature age, makes some wish to put these theorems into the β class, and to leave the emphasis on proof to a later stage.

It is a help at first to draw two parallel lines and then two others crossing them as shown in Fig. 30. Some think it helps if the conditions that a quadrilateral should be a parallelogram are taken first.

There is one important point. Here the boy meets a figure where there are two pairs of parallels, and it is essential that he should associate each pair of alternate angles with the proper parallel lines. He can be made to see the necessity of attaching the names of the parallels involved to each statement of the equality of alternate angles. This practice should be insisted upon. The same point arises later in the usual proof of the intercept theorem.

The various properties of the parallelogram must be learned for use in riders, and it may be well to draw out that the diagonals do not bisect the angles. Such riders will include the properties of the rhombus (more easily seen as a ' diamond ', but not always to be placed thus), with which should be taken those of the kite.

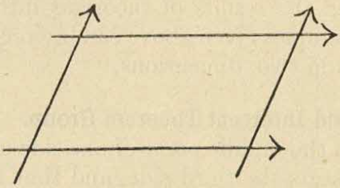


FIG. 30.

Loci. This is not really a group of theorems. The two standard theorems, the locus of P when $AP=PB$, and when the perpendiculars from P to two lines are equal (β), give only poor examples of a most useful idea. At this stage the proofs of these theorems should only be taken as riders in the ' single ' and not ' double ' form. Locus questions can be associated with most groups of theorems. For example, when the parallelogram group is considered the locus of P in Figs. 31 and 32 might be investigated.

In Fig. 31 AD is fixed, $BC=AD$ and $AB=DC$, $BP=\frac{1}{4}BC$.

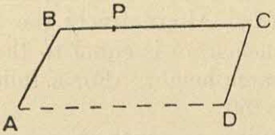


FIG. 31.

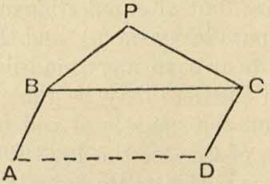


FIG. 32.

Fig. 32 is the same as Fig. 31, except that P is related to BC by means of a triangle BPC of fixed shape and size.

Other examples are :

(a) After ' triangles on the same base and between the same parallels are equal in area ' : what is the locus of P if the area APB is fixed and AB is fixed?

(b) After the mid-point theorem : the locus of the mid-point of OP, when O is fixed and P moves on a straight line or circle.

(c) After Pythagoras' Theorem : the locus of P if AP^2-BP^2 is constant.

(*d*) After the angle property of the circle: the locus of P if $\angle APB$ is constant.

The oddities referred to on page 64 of the 1923 Report are also worth discussing with a class. It is especially important to consider loci in three dimensions; in fact the finding of loci often provides a means of extending the results of theorems into three dimensions, and most of the examples given above can be considered as examples in three, as well as in two, dimensions.

The Mid-point and Intercept Theorem Group. The two theorems, that a line through the middle point of one side of a triangle parallel to another side bisects the third side, and that the line joining the middle points of two sides of a triangle is parallel to and equal to half the third side (α), are both so valuable that they are generally proved separately, and probably before the more general equal intercept theorem. The rider on the parallelograms formed by joining the mid-points of the sides and diagonals of any quadrilateral should not be limited to two dimensions, but should also be shown to apply to any skew quadrilateral, or if the diagonals have been drawn, to a tetrahedron. The intercept theorem (α) should be shown to apply to any number of parallel lines, and so to give the construction for dividing a line into a given number of equal parts. The construction and use of a diagonal scale may also be taken here.

Areas Group and Pythagoras' Theorem. The fundamental theorem of the group is that of equal parallelograms (α). Other results are that of equal triangles (α) and its converse (γ), a triangle half a parallelogram (α), and the construction for drawing a triangle equal in area to any quadrilateral (β). Alternatively the fundamental theorem may be that a parallelogram is equal to the rectangle on the same base and of the same height. For a fuller discussion of this see the 1923 Report, p. 50.

Riders in this group provide practice in a new geometrical method. Pupils are very apt to rely on the use of congruent triangles whenever they have to prove any equality, and so it is necessary to emphasise here that a new method is provided by areas.

For example, the figure in which a line is drawn parallel to a side of a triangle should be dealt with exhaustively by this method of equal areas. It gives a proof of the mid-point theorem and also of the proportional division property. In Fig. 33 in which AD bisects $\angle A$ it can be proved that $BD : DC = BA : AC$ by considering in two ways the ratio of the areas of triangles ABD, ADC. Ceva's theorem can be proved if suitable hints are given.

Useful exercise can be gained by recognising equal areas in a diagram such as Fig. 34, where $AB = CD$, and HC is parallel to GD.

Constructions give considerable scope for the more intelligent boy, while any boy should appreciate them when shown to him, and should be able to follow the proofs. The construction of a rectangle equal to a given rectangle on a different base is suggested by Fig. 35,

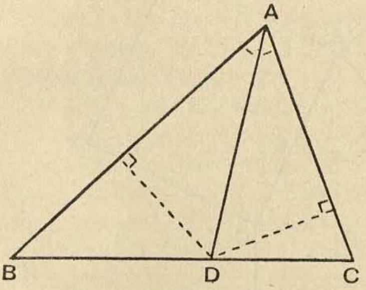


Fig. 33.

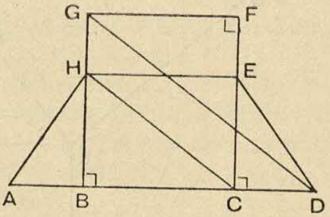


Fig. 34.

ABCD being the given rectangle and *CE* the given base. The construction can also be done by the use of Fig. 36, *ABCD* being the given rectangle and *AE* the given base. There are also the constructions for bisecting a triangle by a line from any point in a side, for bisecting a quadrilateral by a line from a corner or by a line from any point on a side, and the double construction of drawing a triangle with two sides of given lengths equal to a given triangle.

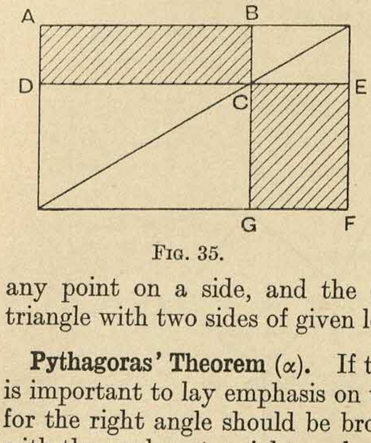


Fig. 35.

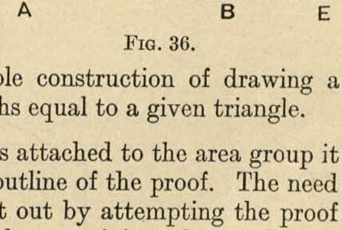


Fig. 36.

Pythagoras' Theorem (α). If this is attached to the area group it is important to lay emphasis on the outline of the proof. The need for the right angle should be brought out by attempting the proof with the angle not a right angle, and by examining where it breaks down. It is interesting at the same time to show, or to give as a rider, the proof by areas of the extension theorems, indicated for the acute-angled triangle by Fig. 37. The corresponding figure for the obtuse-angled triangle will call for some care.

This is a good example of a difficulty which crops up frequently: there are two or more cases of a proposition; in one the figure is easy to follow, in the other not. There is a regular procedure of

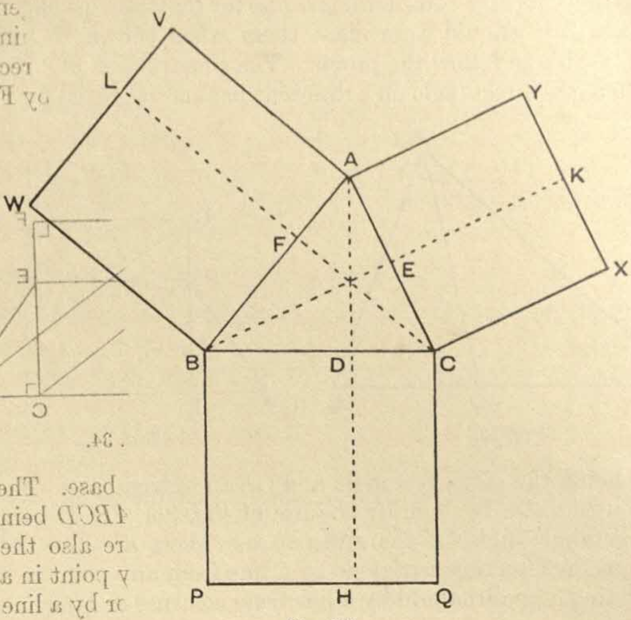


FIG. 37.

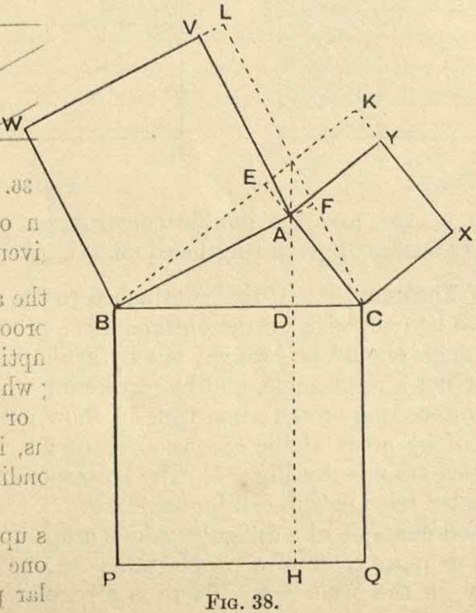


FIG. 38.

great importance which boys should be trained to adopt for themselves. Draw the figure for the easy case and letter it, then with the construction clearly in mind or if necessary written out in front of one, re-draw the figure following the former construction and being careful to retain the original lettering.

Thus in the present instance draw *ABC* with *A* obtuse (Fig. 38), draw squares *BPQC*, *CXYA*, *AVWB*; draw *ADH*, *BEK*, *CFL* perpendicular to the sides.

At this point the change in the appearance of the figure occurs, and we find we have to produce *CA* and *BA*, *XY* and *WV*.

Further examples where two figures are needed are afforded by the equivalence of parallelograms on the same base and between the same parallels, the rectangle property of chords of a circle, and the theorem that the bisector of the vertical angle of a triangle divides the base in the ratio of the other two sides.

Pythagoras' theorem may also be attached to the similar triangle group. Most teachers will probably like boys to work through both proofs and will allow them to choose which they shall be able to reproduce.

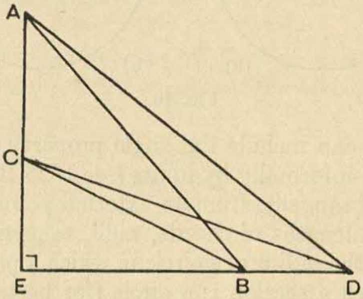


FIG. 39.

To bring out the fact that the theorem is to be used for finding lengths rather than areas, the first exercises will be calculations, which should include questions relating to solids such as cones, pyramids, etc., or to distances between different points of a room. The need is here felt for elementary trigonometry to extend calculations to right-angled triangles in which an angle is given, and this is one of the suitable places for a first introduction of trigonometry.

The other kind of exercise is exemplified by Fig. 39 in which it is to be proved that $AD^2+CB^2=AB^2+CD^2$. This type is useful, as a direct application of the theorem must inevitably give the required results, the only difficulty being to find out how many times the theorem has to be applied. Thus the boy can certainly start the question and will not have to spend time searching for facts that are likely to be useful, and so he should find this an encouraging type of rider.

Proof of the converse theorem follows more easily after boys have had some experience with riders.

The Circle. Symmetrical Chord and Tangent Properties. The theorems of this group are all in the β class, and the results are so easy and expected that this, to the boy, is not a striking group of theorems. Probably it need not be treated separately, but can be fitted in with the work on congruence and Pythagoras' Theorem. As riders on congruence, the theorems are useful illustrations of the right-angled case, and comparison of the two diagrams (*a*) and (*b*) of Fig. 40 helps to show why two sides and a non-included angle are not sufficient for congruence.

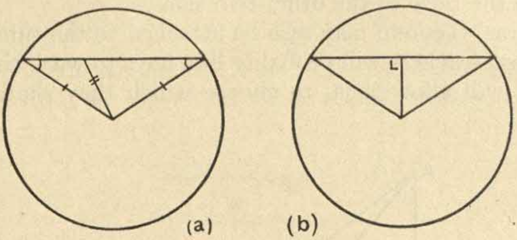
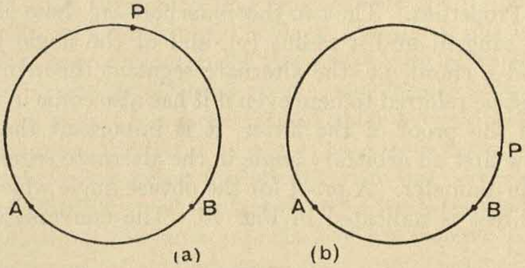

(a) (b)

FIG. 40.

The group also can include the angle property of a tangent and radius (γ), treated informally by limits (see 1923 Report, pp. 44-48), and the lengths of tangents from an external point to a circle (β).

Calculations on lengths of chords, radii, tangents should include the calculation of the radius of a circle in which a plane cuts a sphere, the proof that such a section is a circle (for better boys only), the lengths of tangents to a sphere, and perhaps the volume of a tangent cone. One may also digress here and deal with great circles and small circles of a sphere, equator, meridian, latitude and longitude, and the various distances between two points on a sphere (see p. 107). Constructions in this group include the circumscribed and inscribed circles, their centres being found as the intersection of loci.

Angle Properties of the Circle. The fundamental theorem is that of the angle at the centre (α), and the three deductions are those of the angles in the same segment, (α) the opposite angles of a cyclic quadrilateral, or angles in opposite segments (α), and the angle in a semicircle (α). This is the ideal group ; the results are interesting and are also perfect examples of the deductive rather than the experimental method. Teachers are therefore right in wanting to push on to this group early, as they can give no better model of the deductive process. In the fundamental theorem the boy should be accustomed from the first to draw two circles, to place A and B as fixed points *in the same places*, and then to add P, first near the top, and then near to A or B as in Fig. 41 (a) and (b), and then to complete both figures in the same way. In the quadrilateral theorem boys should be as familiar with the allied result, the exterior angle equals the interior and opposite angle, as with the property of the supplementary angles (compare the allied forms of the angle-sum theorem).



(a) (b)

FIG. 41.

It is also a valuable exercise to use Euclid's proof of this theorem, depending on the sum of the angles in the triangle ABC, Fig. 42, although this method departs from the typical arrangement of a group in which other theorems are each deduced from the fundamental theorem. Other alternative proofs are discussed in the section on Stage C, p. 112.

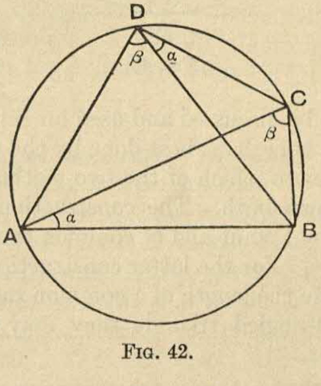


FIG. 42.

Calculations of angles by the use of these properties are needed to drive home the facts, and are generally appreciated by the duller boy as easier than most riders. Boys should be encouraged to mark equal angles by the same small letter as an aid to calculation.

The converse theorems (γ) should be discussed in the search for concyclic points, and it is useful to consider informally converses of the four theorems, discussing whether they are true and whether they are useful. The proofs can be left till later (see p. 93).

Tangent Properties. The two theorems here are those of the angle between a tangent and a radius (γ), and of the angle between a tangent and a chord, i.e. the alternate segment theorem (α). The former must be referred to here even if it has also come in a previous group. In the proof of the latter, it is important that the boy should draw first an arbitrary angle in the alternate segment before drawing the diameter. A proof for the obtuse angle which is useful as being direct is indicated in Fig. 43. The converse (γ) of this theorem should also be discussed and used for riders. The ' constant angle ' construction is perhaps best done by the use of this theorem, though boys differ as to which of the two methods they prefer, and they should be shown both. The construction of a tangent to a circle from an external point and of common tangents to two circles can be included here. For the latter construction it is a help to ask boys first to calculate the length of a common tangent, for in finding the necessary right-angled triangle they may hit upon the construction themselves.

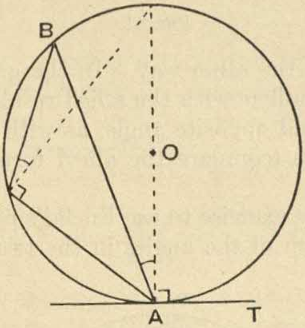


FIG. 43.

The Similar Triangle Group. This section is dealt with more fully on p. 90, where it is pointed out that a first consideration of the group should follow the treatment of congruent triangles. In Stage B the application of the results of theorems to riders is easier, more interesting and of more value than the proofs of the theorems, which are therefore placed in class γ and left till later. When the proofs of the theorems on the similarity of two triangles are discussed, they should be treated as deductions from the theorem of proportional division in one triangle. Hence the fundamental theorem of the group is that a line drawn parallel to the side of a triangle divides the other sides proportionally. The facts that triangles are similar if they are equiangular, or if two pairs of sides are proportional and the included angles equal, follow from the fundamental theorem and its converse ; the third condition for similarity, the sides being proportional, can then be deduced. The fundamental theorem can be proved either from the intercept theorem, or by using the ratio of the areas of triangles of equal areas ; the former is preferred by some who feel that it is a more natural order to develop it from the intercept theorem, the latter by others because it is true for incommensurables and it is easier for boys to write out. This group provides a good illustration of the fact that the fundamental theorem is not, in application, the most important of the group. In Stages A and B general ideas of similarity and the use of similar triangles will precede consideration of the particular property of one triangle with which the fundamental theorem is concerned.

Riders on this section are relatively easy because the method is so definite. If two ratios or two products are to be proved equal, it is necessary first to look for two triangles containing the given lengths as sides, then to prove these triangles similar, and to write down the ratios of the sides, an almost mechanical process.

The fact that the bisector of an angle of a triangle divides the opposite side in the ratio of the other two sides is included in this group. This is an α theorem since the result is unexpected and very nicely shown by deduction. It should be pointed out that this provides a construction for dividing a line either internally or externally in a given ratio.

Rectangle Property of the Circle. This theorem (α) is more easily proved by the use of similar triangles than by other means. Most boys will first prove the results in a set of rectangle property riders on similar triangles. In fact it is almost easier for a boy to prove than to remember the results. This is partly due to the fact that the traditional enunciation includes the words ‘ rectangle ’ and ‘ segment ’, both in unfamiliar senses. Emphasis should be laid on the particular enunciation $OA \, . \, OB = OC \, . \, OD$, each length being measured from the point of intersection O of the two chords, whether it lies inside or outside the circle. The general enunciation might then refer to the product of the segments of the chords, that is to the product of the lengths from the point of intersection to the ends of the chord. Again some boys fail to remember the result because it is left so unnecessarily late in their geometrical course, and so does not become familiar by repetition. Numerical examples, some checked by accurate drawing, are essential to convince the boy of the reasonableness of the theorem. Riders can be found from a triangle with its altitudes, incidentally giving practice in selecting concyclic points; also the construction for the mean proportional, or of a square equal in area to a given rectangle, should be deduced from the theorem.

Some or all of the remaining groups may be omitted if it is desired to lighten the course for the less able boys.

Algebra with Geometry, and the Representation of Algebraic Identities. This is not so much a separate group as manipulative work that is necessary in the course of other work. If AB is a line, a boy must think of AB^2 first as the area of a square, and later as an algebraic expression concerned with the length of AB, and must realise that $AB . BC$ is not AB^2C. For this, in addition to work on Pythagoras' Theorem, examples of the following type are useful: if $ACDB$ is a straight line and $AC=CB$, then $AD-DB=2CD$ and $AD^2-DB^2=2AB . CD$. Such examples are usually done by representing appropriate lengths by single letters. But boys will discover later that this is not always necessary, and will get used to working sometimes with such expressions as $AB . BC$.

The illustration of algebraic identities should be made interesting by considering such products as $a(b+c+d+ \dots)$, $(a+b+c)^2$, as well as the simpler ones. But this may be omitted from the geometry course, and shown instead, as an illustration, in the algebra course to distinguish between $(a+b)^2$ and a^2+b^2.

The Extensions of Pythagoras' Theorem, and the Theorem of Apollonius. These theorems may be omitted by some boys because of their difficulty, but they belong to the α class. The extension theorems are essential when boys in their trigonometry pass the stage of dealing with right-angled triangles, and define the ratios for angles of any size. Then these theorems can be learnt, and the beautiful economy of mathematics exemplified by the fact that the one law, the cosine rule, includes the two rather clumsy results of geometry, as well as Pythagoras' theorem itself, And if a boy does not reach this stage of trigonometry, it is doubtful whether it is worth his while to labour at the results in geometry. It is the general enunciation involving the word ' projection ' that is found difficult. It is a good thing to place this at the end, and treat it as an exercise in composition. After the extensions Apollonius' theorem is pleasantly easy and should be used to calculate a diagonal of a parallelogram when given the two sides and the other diagonal. Riders such as ' prove that three times the sum of the squares of the sides of a triangle is equal to four times the sum of the squares on the medians ' are very useful for the more able boys.

The Concurrence Group. The theorems (α) of this group deal with the concurrence of certain lines in a triangle, viz.

the perpendicular bisectors of the sides,
the bisectors of the angles,
the medians,
the perpendiculars from the vertices to the opposite sides.

For a discussion, see p. 96.

The Inequality Group. The theorems of this group are:

(i) The exterior angle of a triangle is greater than either of the interior opposite angles (α).
(ii) The greater side of a triangle has the greater angle opposite to it, and the converse (α) or (γ).
(iii) The perpendicular is the shortest distance (γ) or (α).
(iv) Any two sides of a triangle are together greater than the third (β).
(v) The theorem about the shortest distance from A to B via a point on a given line (the light-path theorem) (α).

Other results, probably best treated as riders, are

AP increases steadily as P moves round the circle from P_1 to P_2.

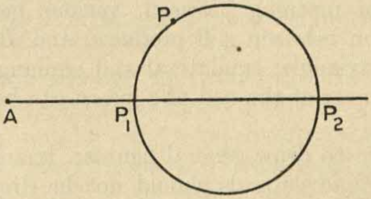


FIG. 44

If P is inside the triangle ABC, $BP+PC<BA+AC$, and $\angle BPC>\angle BAC$.

A detailed discussion of this group is given on p. 99.

The Pentagon Group. This consists of two constructions and one theorem leading up to the construction of a regular pentagon, and a regular decagon.

(i) To divide a line in medial section.
(ii) To construct a triangle with each base angle double the vertical angle.
(iii) If p, d are the sides of a regular pentagon and a regular decagon inscribed in a circle of radius r, then $p^2=d^2+r^2$.
(iv) To construct a regular pentagon and regular decagon.

To these may be added the solution by construction of a quadratic equation.

A detailed discussion of this group is given on p. 103.

VI RIDERS, CONSTRUCTIONS AND STANDARD THEOREMS

The Solution of Riders. The first difficulty which arises when a pupil attempts to solve a rider is the translation of the question into the appropriate figure. To effect this is in itself an achievement of value.

The ability to obtain the figure can be considerably increased by a little concentrated practice now and then. The beginner must be shown how to exact the utmost significance from each word, and his memory must be continually refreshed as to the exact meanings of, and differences of meaning between, various geometrical terms, e.g. the distinction between AB produced and BA produced, between angle and triangle, equilateral and equiangular, and so on. Odd minutes left over at the end of a lesson can be profitably filled in this way.

Boys must learn to draw general figures; triangles must not be drawn isosceles, quadrilaterals should not be drawn as parallelograms; and notice must be taken of any particular specification of the figure, e.g. if it is given that $AB > AC$, the figure should be drawn with this inequality pronounced. It is also necessary to repeat often the warning that things are not necessarily equal because they look equal, and that lines are not necessarily perpendicular because they appear to be so.

A beginning might be made with something like 'ABC is an isosceles triangle right-angled at A; draw the figure', continuing with slightly more complicated figures; it is quite useful to take a few examples from the text book merely for the practice in drawing the figures. Appreciation of geometrical language may also be taught by the converse exercise of drawing a figure on the blackboard, and getting the class to write out suitable descriptions of it, finally obtaining a short and neat description by discussion.

There are, of course, degrees of difficulty in the translation of question into figure, due to the different forms in which questions are stated. In probably the majority of elementary riders the facts are stated in such a way, and in such an order, that the figure can be drawn step by step in the order in which the facts are stated; e.g. ABC is a triangle; $ACPQ$, $BCRS$ are parallelograms outside ABC; QP, SR are produced to meet at O; $ABXY$ is a parallelogram such that BX is equal and parallel to OC; prove that

$$ACPQ+BCRS=ABXY.$$

Here each datum can be taken as it comes and put into effect, without regard to what is to follow; the drawing of the figure is a straightforward exercise analogous to the translation of a succession of independent simple sentences.

But contrast the following: $ABCD$ is a parallelogram such that the bisectors of the angles DAB, ABC meet on CD; prove that $AB=2BC$. Anyone who can do the rider in his head can, of course, draw the figure correctly in the first instance; but a beginner is not in this position. He will probably draw a parallelogram at random, bisect the angles A and B, marking the equal parts as he goes; the bisectors meet at a point which is probably not on CD: why? How ought the parallelogram to be drawn to secure the prescribed result? Conclusion: draw the figure again thus: Draw any line AB; through A and B draw parallels; bisect the angles A and B by lines meeting in E: through E draw the fourth side CD; then prove $AB=2BC$.

Most questions beginning, for example, '$ABCD$ is a quadrilateral', and ending, 'prove so and so about the quadrilateral', are of this type. So, notably, are many questions specifying certain angles of a cyclic quadrilateral and demanding the calculation of others; e.g. $ABCD$ is a cyclic quadrilateral, $EABF$ is a straight line; $\angle EAD=82°$, $\angle FBC=74°$, $\angle BDC=50°$; find the angle between AC, BD.

Here, and in most such cases, it is impossible to follow the question at all without sketching the circle and quadrilateral at random, and putting in the numerical data, which may be violently discordant with the figure as drawn. Having done so it is sometimes easy enough to use the figure, bad though it may be, for the subsequent calculations; but the question may profitably be asked: how is the figure to be drawn to conform with the specification? In fact such a question constitutes a valuable exercise in construction.

The next step in the work of solving riders is to study the figure as it grows. The importance of this should be emphasised. The boy should consider, as he draws each fresh bit, what new fact he knows about the figure, and what are the consequences of each successive piece of information given about it. The facts given, and the deductions from them, should be jotted down or marked in the figure in some distinctive way.[1] It is well that the question should be read again finally so that no point should be missed. Except in isolated cases all the facts given will be needed to prove the required result.

For this stage it is most important that the pupil should be familiar with the main facts of geometry, and should connect them together in groups in his mind. Odd minutes may be used to familiarise the class with the enunciations of standard theorems; these are the rules by which the game of solving riders is carried on.

In the simpler riders, when the process of drawing just described has been followed, it will often be found that the property to be proved has already been discovered. If not, a little thinking backwards from the required property should next be tried. What properties from which the required deduction could be made might the figure possess? Does the figure possess one of these properties? In many cases a clue is to be obtained from the context. The example may be set on a definite group of theorems in the text book. Riders should as a rule be set on groups of theorems rather than on a single theorem.

The solution of a rider depends upon the connection of relevant facts, and spare minutes can be used to great advantage in tracing cross connections between bits of geometrical knowledge. Besides the usual arrangement into groups of interdependent theorems there are other associations in geometry; for example 'right angle' should recall

straight angle,
shortest distance,
sum of angles of triangle or polygon,
allied angles,
theorem of Pythagoras,
line joining centre to mid-point of chord,
cyclic quadrilateral,
angle in semicircle,
tangent and radius for a circle:

while to prove two lines equal we may use

corresponding sides of congruent triangles,
sides of a triangle opposite equal angles,
opposite sides or halves of diagonals of a parallelogram,
equal intercepts,
chords of a circle at equal distances from centre,
chords of a circle subtending equal angles at centre,
equal tangents from an external point.

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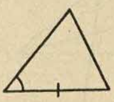
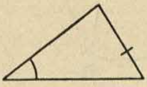
[1] See Appendix 1. Right angles can be marked ∟, and parallel lines by arrowheads.

Similar lists can be made of properties which might be useful in proving two angles equal, or two lines parallel, and so on. In the solution of a particular rider some of these might be specially indicated, others might be obviously inapplicable, inasmuch as they involve the presence of special figures.

In connection with congruence riders, it is a common experience that boys find two equal parts of the two triangles, and fail to find the third parts. Here some drill on the lines of the following table may be useful:

Having found	Look for
2 angles,	any side.
2 sides,	third side, the included angle, or a right angle.
1 side, 1 angle,	
(*a*) 	another angle, or the other including side.
(*b*) 	another angle.

Analysis of the groups of theorems from a different point of view, as in the following example, is helpful.

Consider the tangent group of theorems: enumerate them.

One property needs the centre of a circle, and probably right angles come in.

One requires two tangents from a point to a circle, and leads to two equal lengths, or less often to two equal angles.

One involves two touching circles, and leads to the sum or difference of two lengths, or else a common tangent has to be drawn.

One property involves a tangent and a chord through its point of contact.

This kind of analysis is more suitable for boys who have progressed some way into Stage B.

A useful means of developing a boy's confidence in his own judgment is to give a figure and ask for a list of its properties, reasons being given briefly for the properties discovered. Figures such as Fig. 45 where *ABCD*, *ABEF* are given to be parallelograms; or the figures given on p. 59 are suitable for the purpose.

When a rider takes the form of the construction of a figure to a given specification, the first step is to make a freehand sketch of the kind of figure, and to mark on it the information given in an appropriate way. An idea is gained of the appearance and size of the finished figure. The rough sketch is then examined for a convenient starting point, e.g. a suitable base line, triangle, or circle, from which the rest of the figure could be built up. In some cases the order in which the remainder of the figure can be drawn is then easily seen, as in the example given above of the construction of a parallelogram *ABCD*, in which the bisectors of the angles *A* and *B* meet on *CD*. In other cases additional lines, usually suggested by standard theorems associated with parts of the figures, must be added to the figure before the method of construction becomes plain. For example, if a triangle *ABC*, having given angles, is to be circumscribed to a circle of a given radius, it is evident that in the final figure the circle must be drawn first, and then, from any point on its circumference, a line must be drawn perpendicular to the radius at that point to form the first side of the triangle. How the other two sides are to be drawn is not seen until the radii at their points of contact are drawn in the sketch figure, when it appears that the three radii make angles supplementary to those of the triangle of tangents.

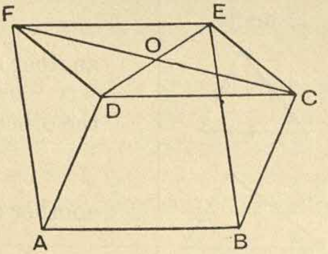


FIG. 45.

In many cases when part of the figure has been drawn the remaining points have to be determined as the intersection of two loci, and the freehand sketch should be examined for a point about which two pieces of information are given, either directly or indirectly; the corresponding loci and the method of their construction must be decided on, and the point can then be fixed in the accurate drawing. For example, in drawing a triangle given its area, vertical angle and base, the base is drawn first, and the position of the vertex with reference to it is determined as the intersection of an arc of a circle (a locus indicated by the magnitude of the vertical angle) and a straight line (a locus indicated by the given area). In more complicated cases there may be more than one way of completing the figure, and it will remain to select as short and neat a method as may be possible. The rough figure should be shown up with the accurate drawing, and should be regarded as part of the work.

It is possible to capture the interest of certain types of less able boys in geometry more easily by means of suitable drawing problems than by any other method.

It is important in rider work to cover as much ground as possible, a large number of easy riders being far more valuable than a few harder ones. It is not necessary for all to be written out, the figures may be drawn and marked, and the proofs given orally, the teacher drawing and marking a figure on the blackboard. A certain number of riders should of course be written out in full.

Another important point is that a particular rider should be looked at from as many points of view as possible: for example, in

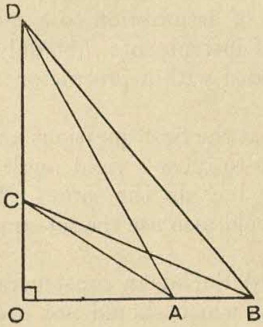


FIG. 46.

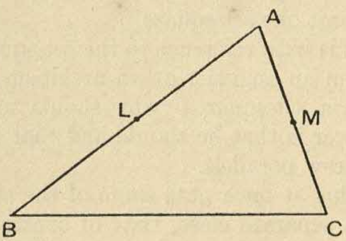


FIG. 47.

Fig. 46 the property $AC^2+BD^2=AD^2+BC^2$ should be thought of as an application of PC^2-PD^2 is constant when P lies on a fixed perpendicular to CD, as well as proved by repeated applications of Pythagoras' Theorem; and in Fig. 47 L, M being mid-points of AB and AC, the parallelism of BC and LM follows both from properties of the parallelogram, and also from those of triangles of equal area on the same base.

Constructions. Geometrical constructions form an exceedingly important part of the course in geometry and, as stated elsewhere, the share of time allotted to them should be not far short of one-third of the whole. They have a special value for those with less aptitude for the subject. It is seldom difficult to find boys whose best achievement in geometry is a genuine and accurate knowledge of how to construct the circumcircle and incircle of a triangle, and some other such constructions, while they have only a half knowledge of theorems and riders.

To some of the standard constructions a certain artificiality belongs, in that they are traditionally to be carried out by the use of ruler and compasses only, while in practice the use of the set-square is more convenient.

Euclid restricted himself to the ruler and compasses, not because they were the only instruments known in his day, but because he wished to build up his geometry on a minimum basis of assumption. They were in fact the embodiment of his postulates, that there is a line joining any two points, that any line has a continuation in either direction, and that there is a circle with any centre and any radius. The easiest way to remember these postulates is to say that we have the use of an ungraduated ruler and a pair of compasses, and that we must not use, in our theoretical geometry, any lines or points which cannot be constructed by means of them.

Our present-day geometry course, on the other hand, is not built only on Euclid's postulates, and lays little stress, with the schoolboy, on the desirability of reducing the basis of assumption to a minimum; and, in our present-day boxes of instruments, not only is the ruler graduated, but the boy is provided with a protractor and at least one set-square.

It is with reference to the set-square that the first questions arise. When an engineer or an architect wishes to draw a right angle he uses a set-square; why should not the boy do the same? The answer is that he should and that he should also use the set-square to draw parallels.

This at once puts some of the standard Euclidean constructions in a separate class, that of constructions which should not necessarily be used in practice.

These constructions are: At a point in a given line to erect a perpendicular to that line. From an external point to draw a perpendicular to a given line. Through a given point to draw a parallel to a given line.

The question arises, should these constructions be omitted from our course of geometry?

Most teachers will, we think, reply in the negative. The constructions afford such convenient illustrations of truths of congruence that they are well worth teaching; in fact, for the second one it is worth while not only to teach Euclid's construction with the principal diagonal of the kite perpendicular to the given line, but also the image method when this principal diagonal lies along the given line, and probably the semicircle method as well.

In teaching these constructions, however, it should be made clear that they are taught for ornament rather than for use, and they need not be emphasised as much as the other constructions. They may very well be taught a little later than has been usual, and after perpendiculars have been drawn with the set-square.

When the parallel or the perpendicular is wanted in drawing a more complicated figure, the use of the set-square should be unrestricted.

There is one construction of this kind which should be and nowadays is usually omitted, viz. Euclid's delightfully ingenious method for drawing from a point C a line equal to AB, which permitted him to transfer distances by means of his compasses.

Should the boy then bisect lines and angles by using his graduated ruler and his protractor? The answer to this as a rule is 'No'. It is generally so much more easy and accurate to bisect lines and angles by the use of ruler and compasses that this should be insisted upon. Of course here it is well not to be pedantic. If a boy is instructed first to draw a 6-inch line, and then to use its middle point in some further construction, it would be silly not to allow the 3″ to be measured as well as the 6″. But, speaking generally, the constructions for bisecting a given angle, or a given line, and for drawing a perpendicular bisector should be perfectly familiar.

As for the construction of a parallel to a given line at a given distance from it, the strictly Euclidean method can be shortened by the use of the set-square, but there is also available the 'eyebrows' method in which a common tangent is drawn by ruler only to two circular arcs whose centres lie on the given line and whose radii equal the given distance. Many people consider this the most accurate and rapid method in practice, so presumably the boy should be allowed to use it, unless instructed to the contrary. He should, however, be able also to do without it.

The construction of a triangle with three given sides, and the special case when the triangle is equilateral, will have been done as very early examples of the use of compasses, and need neither discussion nor proof at this stage.

Thus the only remaining 'elementary' construction is that for dividing a line into equal parts, which should certainly be done, the necessary parallels being drawn by the use of a set-square. The second method, in which only one pair of parallels is drawn, should be known as well as the other, and its convenience on squared paper emphasised.

In the more advanced constructions these elementary ones are taken for granted, so that it is neither necessary to explain their method nor to prove its accuracy, when set to 'construct, explain, and prove' something more elaborate.

A very large number of teachers would be well advised to tackle a greater variety of constructions than they are accustomed to, and to lay more stress on them. For example there is no better way of getting the equality of triangles which are on the same base and between the same parallels into a boy's head, than working away at the constructions for ‘reducing’ a quadrilateral to a triangle, a pentagon to a quadrilateral, etc. To do these properly he must not only know the fact of equality, but be able to recognise how the parallel must run to give the required equivalent triangles, in fact to *see equivalence on a figure imagined before it is drawn*, which certainly carries with it the ability to see the equivalent triangles when they have been drawn.

All these geometrical constructions assist boys to see the true nature of the relations between the different parts of a figure, and thus provide an introduction to logical arguments based on these relations. It cannot be too strongly emphasised that whether or not the construction of a figure is actually performed, one of the best ways of studying the figure is to think how it might be constructed.

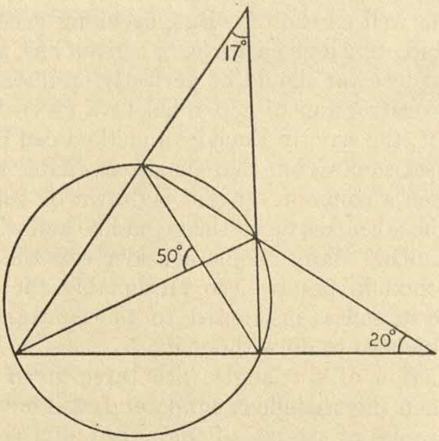


Fig. 48.

A familiar type of question is that which asks for the calculation of elements in a figure on the assumption that others are known ; in some of these cases a further question naturally arises : how can we actually draw a figure to the given specification? This often constitutes a very interesting problem, e.g. given Fig. 48 with the angles as marked, it is not hard to calculate others ; but to construct the figure is probably more interesting and will be found to involve the calculations.

In what follows, remarks are made on the constructions connected with various sections of the course.

Area constructions by equivalent triangles or parallelograms.

(1) The reduction of a quadrilateral to a triangle (mentioned above). This should certainly be a standard construction, with some of its extensions as exercises.

(2) The bisection of a triangle by a line drawn from a point in a side. The bisection of a quadrilateral by a line from a corner, and then from a point on a side, makes a nice exercise as an extension of this.

It is interesting to note that the obvious extension, the bisection of a triangle by a line through an external point, is specialist work, and hard at that, though the bisection of a parallelogram is easy.

(3) To apply to a given line a parallelogram equal to a given parallelogram. This elegant construction, making use of the equality of the complements of the parallelograms about the diagonal of a given parallelogram, is omitted in many of our geometry books, but it is certainly worth including as a rider, even by those who wish to cut down the number of standard theorems and constructions to a minimum (see p. 59).

Circle Constructions. Two of these of especial value are

To draw the circumcircle of a given triangle (or find its centre if drawn).

To draw the inscribed and escribed circles of a given triangle.

If these are done a few times (with differently shaped triangles) *any* boy can become quite firm on them, and then his firm knowledge of the fact that the constructions work can be used

(i) to make sure that he understands the argument by congruent triangles,

(ii) to show him how useful the word ‘ locus ’ is in shortening the long (though easy) argument from congruence.

When we come to the construction of tangents, there is room for some difference of opinion.

All will agree that the boy should learn how to draw the tangent at a given point of a circle, if he is allowed a set-square to make the right angle, but unfortunately the other two constructions for tangents, viz. ‘ To draw a tangent to a circle from an external point ’ and ‘ To draw a common tangent to two circles ’, can be done with greater accuracy by the use of a ruler only, the ruler being laid to graze the circle and perpendiculars being drawn from the centres to show the points of contact, than by more elaborate methods. There is little objection to the semicircle method being learnt for the first of these two ; it is so short and easy, and such an admirable way of bringing home to the boy that the angle in a semicircle is a right angle. Euclid’s method too, though almost forgotten to-day, makes an admirable rider.

Those who aim at the minimum of geometry regard the canonical method for constructing the common tangent to two circles as one of the things which should be omitted.

On the other hand a great advantage of the construction is that it is closely related to the method for calculating the lengths of the tangents. He would be a poor engineer who could not calculate the length of belt needed to drive one wheel from another, and the more theoretical geometer must delight in combining algebraic factors with geometry to express the squares of the common tangents as $AC \cdot BD$ and $AD \cdot BC$ (when the common diameter cuts the circles at $ABCD$).

Those teachers who like coming events to cast their shadows before them should take the opportunity to exhibit and make boys try the 'ruler only' construction shown in the figure for finding the tangents *AL*, *AM* from *A* (Fig. 49). The fact that it cannot be proved by elementary methods makes it attractive to certain types of mind.

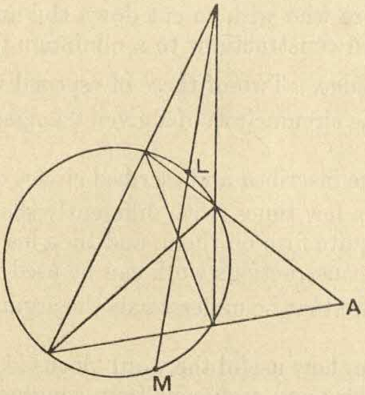


FIG. 49.

Circles to satisfy Given Conditions. Johnson said 'Greek is like lace; everyone would wish to have as much of it as he can get'. The same applies to these constructions, which are so useful in really elegant pattern drawing, especially of continuous curves made up of arcs of circles of different radius, such as occur in architecture. Some of them are by no means easy, and the complete analysis of the 'tangencies' from *PPP* through *PLC* to *CCC* is not elementary work; but teachers who find time for *PPC* ('to construct a circle through two given points to touch a given circle'), or others of similar difficulty, will find that the boys like it.

What could be more instructive in bringing home the relation of tangent circles, than to draw accurately all the circles of a given radius which touch a given circle and one of its chords (produced), especially when there are seven of them?

Constructions involving Proportion and Similarity. The constructions for dividing a line in a given ratio, for finding a fourth proportional and mean proportional, are not difficult and should certainly be included in the ordinary course.

The use of homothetic figures for enlarging or reducing a given figure may very well be of practical use, and is easy and attractive, though to draw a figure similar to one given figure, and equal to another, is too hard except for specialists. As many as possible should construct an example or two of the circle of Apollonius; the elegance of the result will appeal most to those who have first marked out the locus laboriously by finding a series of points.

The construction of a square equal to a given rectilinear figure, being a combination of the reduction of polygon to triangle, the construction of rectangle equal to triangle, and the construction of square equal to rectangle (or mean proportional in a slightly varied form), has already been mastered in theory, and a couple of examples for actual drawing should suffice to bring the method home to the boy.

For the clever boys, including the young mathematical specialist, it is a great mistake to neglect accurate figures. They are a great help in mastering certain types of complicated figures. As examples may be mentioned those connected with the theorems on Pivot-point, Simson's Line (Pedal line), Brocard points, circles round the four triangles formed by four lines, cyclic quadrilateral in which a circle can be inscribed, etc. Such figures may not be demanded as ' constructions ' in the technical sense, but they are among those of which the accurate drawing is of the greatest value.

The teacher who wishes to go into the question of constructions from the logical point of view should refer to *Ruler and Compasses*, by H. P. Hudson (Longmans). The preliminary question is there considered of what can be done with the ruler alone, and what with the compasses alone. It was proved by G. Mohr in 1672 and by L. Mascheroni in 1797 that everything that can be done with ruler and compasses can be done with compasses alone (except, of course, the actual drawing of a straight line); see E. W. Hobson, *Mathematical Gazette*, VII, p. 49.

It has also been shown that all ruler and compasses constructions can be carried out by the ruler and one fixed circle. See Miss Hudson's book already quoted.

Similar considerations are required to show why the classical problems of angle-trisection, quadrature of the circle, and doubling of the cube are impossible by means of ordinary (ruler and compasses) constructions.

Notation. In some early reports of the Mathematical Association it was suggested that the word 'construct' should be employed when a construction *by ruler and compasses only* was intended, and that otherwise the word 'draw' should be used. This distinction was never adopted sufficiently generally to be clear without further explanation by examiners, and the committee feels that it is useless to attempt to get the word 'construct' understood in this sense. If an examiner wishes to bar the use of the set-square, protractor, etc. he should say so. There can be no ambiguity in 'construct using straight-edge and compasses only'.

Standard Theorems. It may be well to explain the sense in which the phrase is used in this report. A *Standard Theorem* is one which is recognised as commonly known, and whose result can be employed in making deductions.

It is not desirable (if it were thought feasible) and it is not feasible (if it were thought desirable) to draw up a definite list of standard theorems. In support of this thesis it may be said that

(*a*) By far the most authoritative list of standard theorems (that of Euclid) was found to be
  (i) too long,
  (ii) inadequate even though so long. Cf. such additions as *Euclid* VI, *B*, *C*, *D* [1] and others such as 'the concurrence of the medians of a triangle'.

(*b*) Attempts have been made to draw up new authoritative sequences in place of Euclid's, but these have not found general acceptance.

(*c*) Any advance by a boy in geometry will usually imply an increase in the number of theorems which he regards as standard, though this increase does not measure his advance, since knowledge and the skill in applying the knowledge need not advance at the same rate.

(*d*) It is always wrong to make a list of what should be known in such a way as to hamper the use of any further knowledge the boy may have. To take an example from algebra, a class may not be expected to know the factors of a^3+b^3, but if a member of it does know them, it would be absurd not to allow him to use them.

Method of Setting Out. In writing out standard theorems careful arrangement and precision of statement should be regarded as necessary. The orderly arrangement of Given, To Prove, Construction, Proof, should be insisted upon. In a few cases part of the construction can be conveniently postponed till it is needed, or the whole construction may be included in the proof.

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[1] *B*, *C* and *D* are respectively:

If the bisector of $\angle BAC$ meets BC at E, then $AB \, . \, AC = AE^2 + BE \, . \, EC$.

If AD is the altitude of a triangle ABC inscribed in a circle of radius R, then $AB \, . \, AC = 2R \, . \, AD$.

Ptolemy's Theorem.

It must not be supposed that this is any real handicap to the pupil, except perhaps to the very slow writer. It is no doubt a difficulty in the first half-dozen theorems which the pupil learns, but once he has got the correct arrangement into his head (which is very soon, if the master insists) *the orderly arrangement is a help and not a hindrance*; the setting down of what is given, and what is to be proved, ensures their being in the boy's mind, and the proof of any theorem, once the gist of the proof is known, can be cast quite easily and naturally into the recognised form.

It is quite a mistake to suppose that to allow slovenly and ill-arranged proofs to pass makes it in any way easier for the boy in the long run. On the other hand the gist of the proof should be known *before* the beginner attempts to cast it into the recognised form.

After a boy has made reasonable progress in geometry he can be set to prepare theorems from the book, but this is a comparatively late stage.

All through Stage B, and, with lower divisions, in a good part of Stage C also, the proofs of theorems, preferably of a group of theorems, should be gone over on the blackboard *before* the class is set to prepare them from the book, the gradual building up of the figure being often an important part of the process. When a proposition is discussed a second time, the diagram should be built up afresh, preferably in a different position.

The normal procedure in which the master elicits the proofs by questions or expounds them, in which the boys have subsequently to prepare the proofs from the text-book or from their own notes, and come into school prepared to write out some one or more of the proofs learnt, has not been superseded, and can only be discarded by the teacher at great risk of his pupils failing to make genuine progress. The best method of preparation is for the boy to try to reproduce the proof without the book.

For a different view, see Appendix 14.

When it was said above that

(i) Euclid's list of standard theorems was too long,
(ii) No limitation should be put to the number of theorems which a boy should regard as standard theorems,

the contradiction was only apparent.

Euclid's list was too long because so many of his theorems were only subsidiary links in one particular chain of reasoning, and of little interest or importance in themselves. It *is* desirable to limit the number of theorems of which a boy may quote the results, by saying that such theorems should have a certain individuality and importance of their own.

An example from congruence of triangles is the following: 'Two triangles which agree as to two sides and a not-contained right angle are congruent', is now universally regarded as a standard theorem. This is justified by the constant occurrence of this special case of the more general theorem, while it would not be justifiable to quote as a standard theorem, 'Two triangles which agree as to two sides and a not-contained angle of four-thirds of a right angle are congruent'.

On the other hand, at a later stage it would be quite justifiable for him to say 'In $\triangle$s ABC, PQR; $a=p$, $b=q$, $A=P$, and the angles B, Q are (for a reason assigned) not supplementary, therefore the triangles are congruent', even though one might not wish to exact a knowledge of the theorem used from a lowest certificate class.

Bookwork and Rider. The setting of a question in two parts, the first being a standard piece of bookwork and the second a deduction, is a commonplace in perhaps a majority of mathematical examinations of all sorts of standards up to, and beyond, university entrance scholarship examinations.

It must not be supposed that the presence of bookwork in more questions—sometimes referred to as 'an increase in the amount of bookwork for which the teacher and pupil are jointly responsible'—really makes the examination any harder. It usually makes the examination easier, though it may make the teacher more ashamed of himself when his pupils fail. The point that must be emphasised is that one of the main functions of the bookwork is that it gives a hint, often a most valuable and necessary one, as to how to do the rider. Thus a boy will very often do *both* bookwork and rider quite successfully in *less* time than he would have taken to do the rider alone in the absence of the bookwork. Those who have attempted to get scholarship candidates to do problem papers will appreciate this point.

Of course this use of the bookwork as a hint is much more important in some cases than others, but it is easy in geometry to name really helpful bookwork. For example:

(i) 'Angle in semicircle is a right angle' before 'altitudes of triangles concur'.

(ii) Mid-point theorem for a triangle before the mid-points of the sides of a quadrilateral form a parallelogram.

(iii) 'Rectangle property of circle' before 'tangents to two circles from a point on their common chord produced are equal ' and so on.

This use of bookwork as a hint may often best be achieved by making the first part of the question ' State the product property of intersecting chords of the circle ', or even ' What are you expected to prove when told to show that *ABCD* is a parallelogram ', rather than ' prove ' so and so ; and this type of bookwork question has rightly been increasingly used by examiners of recent years.

VII FURTHER DETAILS OF TEACHING IN STAGE B

Solid Geometry. The treatment of solid geometry in Stage B will be largely informal and intuitive, with a good deal of both sketching and accurate drawing. Many of the theorems of plane geometry can be extended to three dimensions as they arise. To make this possible, discussion and illustration should make clear the meanings of angles between lines and planes, and should elicit certain facts. These include :

A plane is determined by three points, or by two intersecting or parallel lines.

A plane cuts parallel planes in parallel lines.

Lines perpendicular to the same plane are parallel.

A line perpendicular to a plane at A is perpendicular to any line in the plane which passes through A, and the line perpendicular to two lines at their point of intersection is perpendicular to the plane determined by them.

It should be pointed out that in geometry ' perpendicular ' and ' vertical ' are not synonymous. From their work in Stage A boys should be familiar with the ideas of horizontal and vertical lines and planes.

One of the chief difficulties of the subject is the representation of solid figures on paper. See Appendix 3. At first much use can and should be made of models, but until boys can think in three dimensions and draw convincing pictures to illustrate the problems that arise, little progress can be made. The construction of models takes much time ; in any but the simplest cases a boy must have a clear mental picture of the problem before he can begin to construct the model, and if he is able to represent his visualisation in a sketch he will be able to deal with it as easily without a model as with one. For example, in the construction of a skew pyramid to a given specification the net of the pyramid must first be drawn ; if a boy can do this correctly, elaboration with cardboard, scissors and paste adds little of value. If however the net is drawn incorrectly, the best way to bring home the error to the culprit will be to let him cut it out and try to fold it into the required solid.

In many cases a model extemporised from the material at hand, books, set-squares and pencils, will be useful in developing the power of seeing figures in the solid.

For demonstration purposes the apparatus described in the *Gazette*, Vol. XII, p. 286, may be used. Sketches on the board should accompany the use of the apparatus.

Much of the work in solid geometry should take the form of riders upon the theorems of plane geometry: a selection of these follows, arranged under the proposition-groups.

Riders in Solid Geometry. *Note.* In the following examples the expression 'the cuboid $ABCDEFGH$' means the cuboid with opposite faces $ABCD$ and $EFGH$ joined by edges AE, BF, CG, DH. It is recommended that this notation be standardised; questions can then be set concisely without ambiguity.

Congruence Group

1. AB, CD are two equal lines not in the same plane. If $AC=BD$, prove that $\angle BAC=\angle BDC$. If the triangle ABC is rotated about BC, must A pass through D?

2. The height of a right pyramid on a square base is equal to half the diagonal of the base. Prove that the faces are equilateral triangles.

3. $ABCDEFGH$ is a cube; prove that $\angle ACH=60°$.

4. $ABCDEFGH$ is a cuboid; prove that $\angle FCH=\angle AHC$.

5. $VABCD$ is a right pyramid on a square base; prove that perpendiculars from A, C to VB intersect.

6. P is any point on the side AB of a triangle ABC. The triangle is rotated about BC; A', P' are the new positions of A, P; prove that $\angle P'AC=\angle PA'C$.

7. P, Q are two points on the perpendicular bisector of AB; at P a perpendicular PR is erected to the plane of PQ, AB. Prove that the triangles RAQ, RBQ are congruent.

8. $ABCD$ is a square whose diagonals meet at O; OP is perpendicular to the plane of the square; PA, PB, PC, PD are joined. Point out the various sets of congruent triangles and prove them congruent.

9. $ABCD$ is a rectangle; at N, a point in its plane, a perpendicular NP to the plane is erected. P is joined to A, B, C, D. Point out pairs of congruent triangles (i) if N is the mid-point of AB, (ii) if N lies on the line joining the mid-points of AD, BC at the point of trisection nearer AD. Give a proof in each case.

10. OA, OB, OC are three given lines not in the same plane. Show how to find a line OP so that $\angle AOP=\angle BOP=\angle COP$.

Parallelogram Group

1. Show that the diagonals of a cuboid meet at a point.

2. P, Q are points on the edges AB, GH of a cuboid $ABCDEFGH$ such that $AP=GQ$. Prove that $AQ=PG$.

3. $ABCD$ is a parallelogram; P is any point not in its plane; Q is the mid-point of PB; AQ is produced to R, so that $AQ=QR$. Prove that RD bisects PC.

4. $ABCD$, $ABEF$ are two parallelograms in different planes; prove that DF is parallel to CE.

5. Three lines AOB, COD, EOF, not in the same plane, meet at O which is the mid-point of each. The parallelograms $AECP$, $BFDQ$ are completed. Prove that $PF=EQ$.

Loci

Find the locus of

1. A point at a given distance from a given point.

2. A point at a given distance from a given straight line.

3. A point equidistant from three given points.

4. The mid-point of a straight rod whose ends slide on the inside of a hollow sphere.

5. The centre of a sphere which rolls on a plane.

6. The centre of a sphere which rolls on the inside of a hollow cone.

7. A straight line which passes through a given point in a plane and makes a given angle with the plane.

8. The axis of a cone which rolls on a plane.

9. The axle of a garden roller which is pulled up over a step.

10. The common edge of two adjacent faces of a brick which slide on two thin fixed parallel rods.

Mid-Point and Intercept Theorem Group

1. $ABCD$ is any quadrilateral; it is folded about BD so that ABD, CBD are in different planes. Prove that the mid-points of AB, BC, CD, DA form a parallelogram.

2. The diagonals of a cuboid $ABCDEFGH$ meet at O; any straight line XOY through O meets the planes $ABCD$, $EFGH$ at X, Y. Prove that $AX=GY$.

3. $ABCD$ is a square; V is any point not in its plane; L, M, N, P are the mid-points of VA, VB, VC, VD. Prove that L, M, N, P are coplanar and are the corners of a square.

4. At the corners of a square horizontal court, vertical poles of heights 12′. 10′, 6′, 7′ are erected in order round the court. The

tops are joined diagonally by straight wires. Will the wires intersect? If not, what alteration in the height of the shortest pole is necessary to make them do so?

5. Prove that the lines joining the mid-points of opposite edges of a tetrahedron meet in a point.

Pythagoras' Theorem

1. The edges of a cuboid are 3′, 4′, 12′ long: find the length of a diagonal.

2. A piece of wire is bent into three parts AB, BC, CD, each of the outer parts being at right angles to the plane of the other two; $AB=12''$, $BC=6''$, $CD=12''$. Find the distance of A from D.

3. A pyramid of height 8″ stands on a square base each edge of which is 1′. Find the area of the sloping faces.

4. What is the radius of the sphere circumscribing a cuboid $2'' \times 3'' \times 6''$?

5. $ABCD$ is a rectangle; $AB=6''$, $BC=8''$. It is folded about BD so that the planes of the two parts are at right angles. Find the new distance of A from C.

6. Two concentric circles are drawn whose radii are equal to the height and the slant height of a cone respectively. Show that the area of the ring between them is equal to the area of the base of the cone.

7. Four equal spheres, radius 1″, are placed on a table so that each touches two others and their centres form a square. A fifth equal sphere rests on them. Find the height of its centre above the table.

8. A cubical box, edge 4′, contains a sphere of radius 1′. Find the radius of the largest sphere that can be fitted into the box at the same time.

9. A circular table 1′ in diameter has three legs attached to its edge at equal distances; each leg is 5′ long. It stands on level ground, the points of contact of the legs with the ground forming an equilateral triangle. Find the side of the triangle if the table is $4\frac{1}{2}'$ above the ground.

Circle: Symmetrical Properties

1. A sphere with centre O is cut by a plane: ON is the perpendicular to the plane. Show that the curve of intersection of the sphere and plane is a circle with centre N.

2. A lens in the form of a segment of a sphere is 6″ in diameter and 1″ thick. Find the radius of the sphere of which it is part.

3. A sphere, radius 1″, rests in a hollow inverted cone whose

vertical angle is 60°. Find the distance of the centre of the sphere from the apex of the cone.

4. Prove that the curve of intersection of two spheres is a plane curve by determining the plane in which lie all the points of intersection.

5. A rectangular plate $3'' \times 4''$ rests horizontally in a hemispherical bowl 13″ in diameter. What is the height of the centre of the plate above the lowest point of the bowl?

Angle Properties of the Circle

1. A, B are fixed points. What is the locus of P if (i) $\angle APB$ is a right angle; (ii) if $\angle APB$ is constant but not a right angle?

2. A, B, C are three fixed points on a sphere. What is the locus of a point P on the sphere such that $\angle APB = \angle ACB$?

3. A, B are fixed points on a sphere such that AB is equal to the radius; P is another point on the sphere such that $AP = BP$. What are the greatest and least sizes of $\angle APB$?

4. A, B, C are three points on a sphere such that $AB = BC$ and $\angle ABC = 30°$. The perpendicular from the centre of the sphere on to the plane ABC meets it at P. Prove that $PB = AC$.

5. A circular hole centre O radius 1″ is cut in a thin horizontal plate. The 30° corner of a thin set-square is passed through the hole from below. Show that it is possible to reach with this corner all points inside a hemisphere centre O radius 3·73″. Can one reach further?

Tangent Properties

1. Prove that all the tangents to a sphere from a given external point are equal.

2. S is a sphere of radius a, centre O; C is a circle centre O and of radius $b\,(b > a)$. Prove that all the tangents to S from any points on C are equal, and express their length in terms of a and b.

3. A cone C touches a sphere S in a circle K; K' is another circle on C. Prove that the tangents to S from points on K' are all equal.

4. Two spheres S, S' touch a cone C in circles K, K' and do not touch or intersect each other. A plane P between S and S' touches them at F, F' and cuts C in a curve E. Prove that the sum of the distances of any point on E from F, F' is constant and equal to the distance between K and K' measured along any line through the apex of C.

5. A sphere rests in a triangle of rods. Prove that the perpendicular from the centre of the sphere on to the plane of the triangle meets it at the incentre.

6. A, B are fixed points on a sphere. AT is any line in the tangent plane at A. What is the locus of a point P on the sphere such that $\angle APB = \angle BAT$?

Similar Triangle Group

1. A rod 2′ long is held horizontally 3′ above the floor. What is the length of its shadow if the lamp is 8′ above the floor?

2. A sphere rests in a hollow inverted cone with its centre 5″ from the apex of the cone. The height of the cone is 8″ and its diameter is 1′. Find the radius of the sphere.

3. Prove that the section of a pyramid on a triangular base by a plane parallel to the base is a triangle similar to the base.

4. $ABCDEFGH$ is a cuboid; $EF = 6''$, $FG = 10''$. A rectangle $PQRS$ is formed in the plane of $EFGH$ by lines parallel to EF, etc. and 1″ outside them, P corresponding to E, etc. If PA, QB, RC, SD are produced, will they meet in a point?

5. $PABC$, $QABC$ are two pyramids on the same base ABC and PQ is parallel to the plane of ABC. Prove that the sections of the pyramids by a plane parallel to ABC are congruent triangles.

6. A rectangular table 5′ × 8′ and 3′ high stands on a level floor under a hanging lamp. The shadow on the floor of the shorter side is 8′ long. Find the length of the shadow of the longer side and the height of the lamp above the table. Does it matter where the lamp is?

7. A cuboid 2″ × 3″ × 4″ rests with one of its largest faces on a table. A hollow cone, whose height is equal to its base-diameter, rests on the table covering the cuboid and touching its four upper corners. Find the height of the cone.

Rectangle Properties of the Circle

1. A sphere of diameter 6″ rests in a 5″-diameter hole in a table. A thin square plate of side 10″ with a central circular hole of diameter 2″ rests with one edge AB on the table and the circumference of the hole lying on the sphere. Find the distance of AB from the centre of the 5″ hole.

2. Three lines OAB, OCD, OEF not in the same plane cut a sphere at A, B, C, D, E, F. Are the triangles ACE, BDF similar?

3. (Spherometer.) The vertices A, B, C of an equilateral triangle lie on a sphere; the diameter PQR of the sphere perpendicular to the plane ABC cuts it at Q. If $PQ = 1''$ and $AB = 6''$ find the diameter of the sphere.

Extensions of Pythagoras' Theorem and Theorem of Apollonius

1. $ABCD$ is a rectangle and P a point not in its plane. Prove that $PA^2+PC^2=PB^2+PD^2$.

2. $ABCD$ is a tetrahedron: $\angle BAC=\angle CAD=\angle DAB=1$ right angle. Prove that BCD is an acute-angled triangle.

3. $ABCD$ is a rectangle whose diagonals meet at O; P is any point not in its plane. Prove that

$$PA^2+PB^2+PC^2+PD^2=4PO^2+AC^2.$$

Inequalities

1. A, B, C are fixed points; whereabouts may P lie if
   (i) PA is greater than PB?
   (ii) $\angle PAB$ is greater than $\angle PBA$?
   (iii) PA makes a greater angle than PB with the plane ABC? Illustrate this with a sketch and give reasons.
   (iv) $\angle APB$ is greater than $\angle ACB$?
   (v) the perpendicular from P to AB is greater than the perpendicular from P to AC?

2. Whereabouts may P lie if the perpendicular from P to the plane $ABCD$ is greater than the perpendicular from P to the plane $ABEF$, P lying in the wedge between the half-planes?

3. P and Q are two points not in the plane ABC but on the same side of it. Determine the point X of the plane so that $PX+XQ$ is a minimum. Determine the point Y of AB so that $PY+YQ$ is a minimum.

4. A rectangular sheet of paper $ABCD$ is folded about BD so that AC is now equal to AD. Show that $\angle BAC$ is still acute.

Similarity. It has already been suggested that some attention should be given to similarity in the course of the Stage A work. We further recommend that it should be introduced in Stage B a good deal earlier than has been customary. If this is done, the scope of the work will be widened at certain points, and similarity will no longer be left as a detached section to be treated at the end of the course if there is time.

The most natural time to begin this work is immediately after congruent triangles have been discussed and used. Having seen that the conditions for congruence are an outcome of the methods used for constructing triangles, it is easy to deal with similar triangles in the same way, and to use them for numerical applications followed by other riders.

It will be necessary, however, to devote one or two preliminary lessons to the study of similarity in general, before dealing with the particular case of similar triangles. This lesson should be merely a revision and extension of Stage A work, and in any case it will be a treatment rather on Stage A lines.

We may well begin in three dimensions, by discussing, for example, a model of a ship. Any boy will see at once that it is necessary for the masts of the model to be not only the correct length, but set at the correct angle. The question of the length leads to simple arithmetical exercises, and in the course of these it should be made clear that the relations may be expressed in either of two ways. Not only

$$\frac{\text{length of mast of model}}{\text{length of mast of ship}}=\frac{\text{length of model}}{\text{length of ship}}$$

but also

$$\frac{\text{length of mast of model}}{\text{length of model}}=\frac{\text{length of mast of ship}}{\text{length of ship}}.$$

It is also worth pointing out at this stage that the sail-areas and weights are not in the same ratio as the lengths.

The transition to plane figures may be effected by reference to maps. If we try mapping a four-sided field by means of its four angles, it will soon be agreed that the maps produced that way are not all 'the same shape', in spite of being equiangular. The sides must be proportional as well. The question may also be asked whether we could work with the sides only, and a meccano model may be used to show that by settling the lengths of the sides we do not settle the shape. The word 'similar' should now be defined precisely, and we may illustrate it by the experiment of drawing inside a given rectangle a smaller rectangle similar to the original. We may then proceed to discuss the special importance of the triangle in mapmaking, and the geometrical property to which this is due.

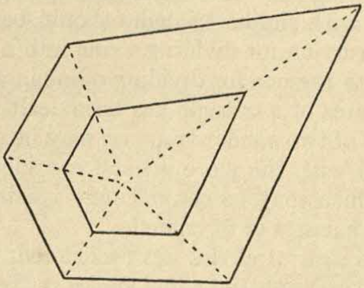

FIG. 50.

If further time is available, the 'spider-web' construction for enlarging a figure (Fig. 50) provides a pleasant and profitable way of using it.

After this preliminary work, the next stage is to consider the construction of a triangle similar to a given triangle, and to tabulate the sets of data necessary to determine the shape. This leads to the sufficient conditions for similarity, which may be placed alongside those for congruence. They are then used for numerical exercises and other riders.

In this work it is advisable that similar triangles should be lettered with their vertices in corresponding order, the corresponding angles having first been marked in a figure. This ordering of the letters is desirable not so much because it may be used for obtaining the equal ratios in a mechanical way, but rather because it draws attention to the exact manner of correspondence, and through that facilitates the correct statement of the equalities of ratios.

When the similarity of two triangles has been established with a view to using equal ratios, there is a considerable advantage in writing down all three ratios :

$$\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}.$$

The selection of the useful pair is then easier than when done from the figure. But boys should also learn to write down the ratios in the way used in trigonometry :

$$\frac{a}{b}=\frac{a'}{b'},\ \frac{b}{c}=\frac{b'}{c'}.$$

Further points at which the work naturally leads on to consideration of similarity are as follows :

(i) After the theorems on the line bisecting two sides of a triangle, we may consider the more general case of a line dividing two sides in the same ratio, discovering the facts by using the intercept theorem, and ignoring the 'incommensurable' case. The connection of this figure with similar triangles should be noticed. In like manner the construction for dividing a line into a given number of equal parts leads to the one for dividing a line in a given ratio.

(ii) When the area of a triangle has been dealt with, the relation between the areas of two similar triangles may be discussed, and the result coordinated with the more general consideration of areas of similar figures, which may be convincingly treated by means of a network either of squares or of triangles.

(iii) As soon as arcs of circles are considered, it is necessary to assume that equal arcs subtend equal angles at the centre. We may as well make the corresponding assumption for unequal circles, namely that arcs which subtend equal angles at the centres are proportional to the radii. As a special case, circumferences are proportional to diameters, and we obtain the formula $C=\pi d$.

(iv) After the group of theorems on angles in a circle, it will be easy and natural to take those dealing with the product of segments of a chord, proving them by means of similar triangles. The algebraical nature of these results may be used as an excuse for recalling Pythagoras' theorem, and this also may be proved by similar triangles before other algebraical results—such as the extensions of Pythagoras' theorem—are taken.

The important thing in the early treatment of similarity is to have plenty of exercises and riders done. As suggested on p. 64 it will be convenient to consider formal proofs of the theorems, taken as a group, at a later stage.

Converses. A theorem in geometry consists of a statement that a conclusion may be drawn from certain independent suppositions. Theorems converse to it are obtained by interchanging the conclusion with any one of the independent suppositions, and if the original theorem is true, the converse theorems may be true or may be false.

In elementary geometry some theorems are based on only one supposition, and so have only one converse.

As simple examples we may give:

If in a triangle ABC the angle A is obtuse, BC is the longest side. The converse is not true.

If the opposite sides of a quadrilateral are parallel, a diagonal divides it into two congruent triangles.

The converse is not true, for if a diagonal divides a quadrilateral into two congruent triangles, the quadrilateral may be either a parallelogram or a kite.

Examples of a theorem with two converses are easy to find.

'Triangles on equal bases and having equal altitudes are equal in area' gives as converse theorems,

'Triangles of equal area on equal bases have equal altitudes', and

'Triangles of equal area having equal altitudes have also equal bases'.

The alternate segment theorem is:

If (1) AB is a chord of a circle, and (2) AT touches the circle, and (3) P is a point on the circle on the side of AB remote from T, then (4) the angle BAT=the angle APB (all the points being in the same plane, which is not treated here as a separate datum).

This has three converses:

Converse 1. If AT touches a circle through A and P, and if equal angles are made, TAB on the same side of AT as P, and APB on the same side of AP as T, then B must lie on the circle.

Converse 2. If AB is a chord of a circle APB, and if AT is drawn on the side of AB away from P, making the angle BAT equal to the angle APB, then AT must touch the circle.

Converse 3. If AT touches a circle through A and B, and if at P, on the side of AB away from T, the angle $APB =$ the angle BAT, then P must be on the circle.

The former of these two examples is often stated in the form 'Triangles on the same base and between the same parallels are equal in area'. It is worth noticing that 'between the same parallels' implies 'equal altitudes' and 'on the same side of the base', so that one converse theorem is 'Triangles of equal area on the same base *and on the same side of it* have equal altitudes (i.e. are between the same parallels)'.

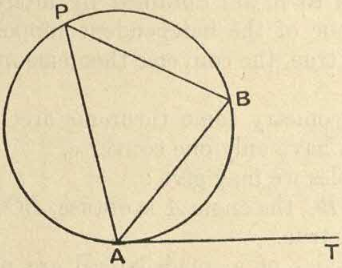

FIG. 51.

When the boy learns the theorems on angles in a circle, he will need to use the converses for establishing that four points are concyclic. Here is an excellent opportunity for discussing the existence of converses. For example in one case of the fundamental theorem two facts are given and a third is to be proved.

Given (1) O is the centre of a circle,

(2) C is a point on the major arc AB.

Prove (3) $\angle AOB = 2\angle ACB$.

It will be brought out by informal discussion that if (1) and (3) are interchanged the converse is not true. If (2) and (3) are interchanged the converse is true *provided O and C are on the same side of AB*. This necessity for stating explicitly in the converse some fact which was implicit in the original theorem, has already been noted in the converse of the theorem of triangles of equal area.

All the possible converses of the angle properties should be discussed in this way, and so the conditions for concyclic points obtained. These may then be used for rider work, before the formal proofs are considered.

This example illustrates the principle that in dealing explicitly with the converses of any theorem the following is the correct order of procedure:

(1) an informal discussion on the converses,

(2) the proofs of the converse theorems.

In some cases in elementary geometry a theorem and its converses are such that each may be proved, independently of the others, directly from previous theorems or assumptions. In these cases the theorems are often of about equal importance, and one does not claim precedence over another. Such theorems are usually taught side by side as members of a group. It is best to adopt the side by side method where possible so as to avoid the question of order. As examples we may give

(1) In a triangle ABC
  (*a*) if $AB=AC$, then $\angle ACB=\angle ABC$,
  (*b*) if $\angle ACB=\angle ABC$, then $AB=AC$.

(2) (*a*) The perpendicular from the centre of a circle to a chord bisects the chord.
  (*b*) The line joining the centre of a circle to the mid-point of a chord is at right angles to the chord.
  (*c*) The perpendicular bisector of a chord passes through the centre of the circle.

Again a theorem will sometimes be proved and then used in a direct proof of its own converse; Pythagoras' Theorem is an example. There is in this case a definite order for the two theorems.

A third method of proof is by *reductio ad absurdum*. This method will probably be introduced early, for example to prove that triangles of equal area on the same side of the same base lie between the same parallels. As the 1923 Report pointed out, the method is fundamental in the proof of converse properties. It consists essentially in proving a property true by showing that all other possibilities lead to contradictions and therefore are false. The difficulties for the learner are

(*a*) to get the nature of the argument clear in his mind,

(*b*) to express the argument clearly in words.

Bound up with (*a*) is the difficulty that 'the figure has to be drawn wrong'. Boys who are accustomed to drawing freehand figures find much less difficulty over this point, than those who habitually use ruler and compasses. The difficulties of this method are considerable, and it is debatable whether proofs by *reductio ad absurdum* should be postponed till Stage C *a* or made less formidable by earlier introduction, repetition and slowly growing familiarity. It can be maintained that, as in more advanced work, the effort required for a boy to write out clearly one of these proofs helps him to understand the principle of the proof, and that therefore such work should not be postponed too long.

The Concurrency Group. The theorems that prove the concurrence of various sets of three lines connected with a triangle form an obvious group, and at the same time furnish an excellent example of geometrical results which can be linked together in more ways than one.

The concurrence of the perpendicular bisectors, and of the angle-bisectors, will presumably have been two of the earliest riders on

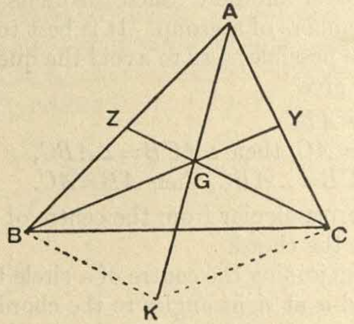

FIG. 52.

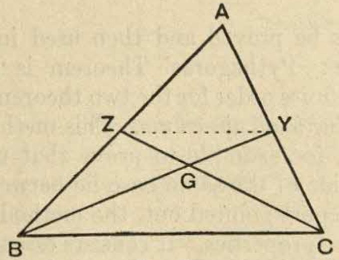

FIG. 53.

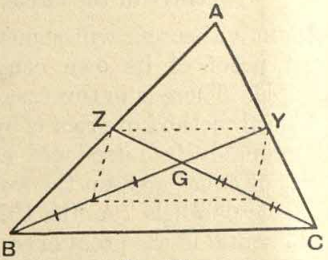

FIG. 54.

the locus theorems, and they are needed in connexion with the construction of circles. The concurrence of the medians will naturally have been proved as a standard result after the mid-point theorem. Three different proofs are suggested by figures 52, 53, 54 ; in Fig. 53 the similar triangles ZGY, CGB are used.

The perpendiculars from the vertices to the opposite sides (loosely but conveniently called the ' altitudes ') may be proved concurrent by showing that they are the perpendicular bisectors of the sides of the triangle PQR in Fig. 55. Alternatively if the perpendiculars BE, CF meet at H (Fig. 56), it is easy to show that AH produced cuts BC at right angles. For this purpose the boy must see and use the circles on BC, AH as diameters, and he can have no real grasp of the properties of this important figure without the ideas involved in this proof.

All four of the results will have been found out by the beginner in some of his earliest use of ruler and compasses. It is good policy at the present stage to link together in the boy's mind the four results and their proofs. A natural way to do this is to revise the first two results just before taking the last two. And if the proofs of Figs. 52, 56 are used, it can be pointed out that the same method may be adopted for all, viz. draw two of the lines, join their point of intersection to the third point, and prove that this joining line is the third line required.

The group is thus completed, but it is both interesting and instructive to discuss various other proofs, the relations between the four results, and extensions to the tetrahedron.

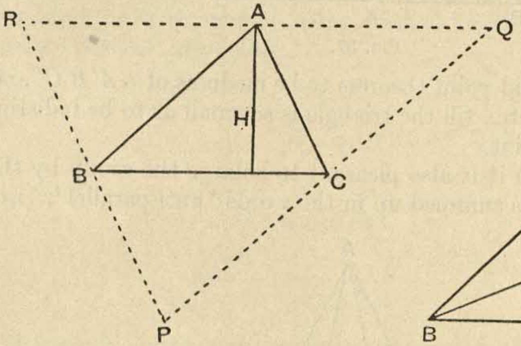

FIG. 55.

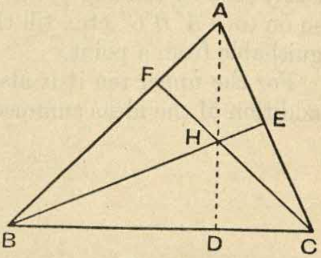

FIG. 56.

The change of triangle from *ABC* to *PQR* in Fig. 55 should be compared with the results connected with the e-centres. If, in Fig. 56, *DEF* is taken as the original triangle, *A*, *B*, *C* are the e-centres.

It should be noticed that the proofs of the median result by Figs. 53, 54 make use of two medians only, with a judicious use of the word 'similarly'. This process can be pushed a step further, and the proof be made to depend upon *one* symmetrically defined point, though this kind of work is only suitable perhaps for the upper 40 per cent. of our pupils. For example if the concurrence of the altitudes is taken before that of the medians and if the proof of Fig. 55 is used, it follows from the similarity of the triangles *ABC*, *PQR* that $AH=2OX$. See Fig. 57. Hence by the similar triangles *AGH*, *XGO*, the median *AX* passes through the point one-third of the way from the circumcentre to the orthocentre. This line of collinearity of *O*, *G*, *H*, also contains the nine-point centre. See p. 162.

The vector proof $\overline{VA}+\overline{VB}+\overline{VC}=\overline{VA}+2\overline{VX}=3\overline{VG}$, where X is the mid-point of BC and G divides AX in the ratio 2 : 1, is another symmetrical proof.

Demonstration by ' infinite descent ' is also perhaps only for the upper 40 per cent. though it is hard to find a boy who is not interested in it. It applies to the medians AA', BB', CC'. These are easily seen by the mid-point theorem to be medians of $\triangle A'B'C'$ and so on to $\triangle A''B''C''$ etc., till the triangle is so small as to be indistinguishable from a point.

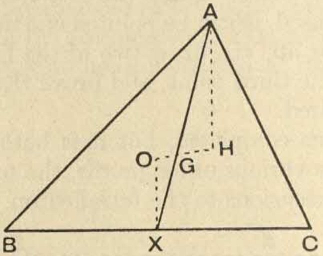

FIG. 57.

For the upper ten it is also pleasant to enlarge the group by the addition of the ideas summed up in the words ' anti-parallel ', ' isogonal ', and ' anticentre ', and to prove the theorem that if three lines through A, B, C concur, so also do their isogonals. This only needs $PM \,.\, P'M'=PN \,.\, P'N'$ in Fig. 58. One then has the results that the orthocentre is the anticentre of the circumcentre, and the incentre is its own anticentre. To complete the group, the symmedian point is introduced.

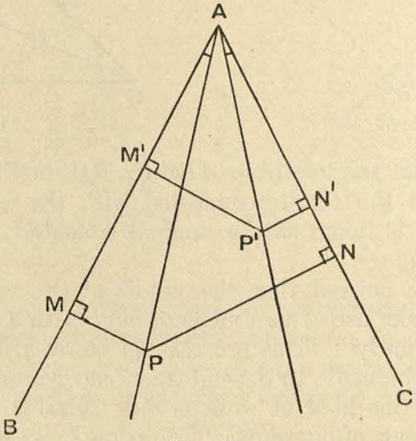

FIG. 58.

Another interesting consideration for the teacher is that three of the four results deal with lines through the vertices and are special cases of Ceva's result, that $\Pi \frac{BX}{XC} = +1$ is the condition for the concurrence of AX, BY, CZ. But the concurrence of the perpendicular bisectors is in another category. It follows from the condition, $\Sigma(BX^2 - XC^2) = 0$, of concurrence of perpendiculars to the sides at X, Y, Z. This condition also proves the concurrence of the 'altitudes'.

The Inequality Group. It is a curious point in the history of mathematical teaching that, at the time when the inequality is being regarded as of increasing importance in the teaching of mathematical specialists, and being more and more stressed, the early training in dealing with inequalities, provided for all in elementary geometry, is in danger of being pushed to the background, and possibly omitted altogether.

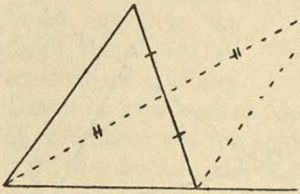

FIG. 59.

There is no doubt that among many teachers the inequality group of theorems is out of fashion and apt to be neglected. The reasons for this neglect appear to be the following.

(i) The properties proved are all rather obvious, and at least one of them (the perpendicular is the shortest distance) has been long assumed in virtue of informal discussion in Stage A.

(ii) The riders on the theorems are found to be hard.

(iii) These theorems are isolated, and can be omitted without the loss being felt elsewhere.

To counter these objections, it will be necessary to take the theorems in some detail.

The fundamental theorem is that the exterior angle of a triangle is greater than the interior opposite angle.

This theorem is not in the same danger of being neglected as the rest of the group.

It is noticeable, however, that this theorem, whose truth is independent of the parallel postulate, is nowadays usually brought to the beginner's notice as a corollary to the theorem that the exterior angle of a triangle is equal to the sum of the two interior opposite angles, whose truth does depend on the parallel postulate.

Hence it is particularly desirable to retain Euclid's proof, from Fig. 59, as a rider—it is a good rider on congruent triangles.

The next theorem is that the greater side of a triangle has the greater angle opposite to it.

This result might be classified among the fairly obvious, though certainly not among the quite obvious. It is, however, a truth which should be known, and should therefore be got to the forefront of the pupil's consciousness, *either* by an informal stage A treatment, *or* by the formal proof.

If this alternative is agreed, the case for the formal proof is almost won. It is quite as easy as and much more convincing than an informal one. The formal proof also introduces the important form of argument *a fortiori* which all boys should know (but many don't).

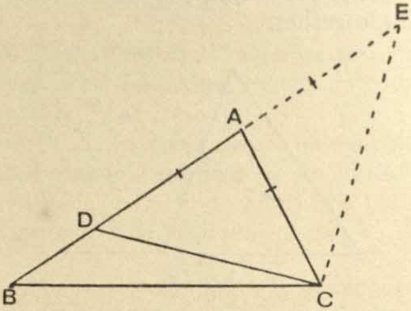

FIG. 60.

Moreover the figure used (especially if amplified by the dotted lines, to show its connection with the fact that two sides of a triangle are greater than the third) is one of the important, because often recurring, figures of geometry ; for

(i) it serves this theorem,
(ii) it comes in for the theorems about the bisector of the vertica angle,
(iii) it is used to construct a triangle given the base and sum or difference of the other sides,
(iv) it shows $\frac{1}{2}(C-B)$ as well as $\frac{1}{2}(C+B)$, and is used to construct a triangle given the base and the difference of the base angles,
(v) a perpendicular to CD at D gives a good proof of the formula for $\tan\frac{1}{2}(C-B)$.

The proof of the converse theorem that the greater angle of a triangle has the greater side opposite to it introduces the important type of argument ' proof by exhaustion '. This should be known and there is no better place for learning it than here.

The alternative direct proof (using 'two sides greater than the third') is easy and worth attention, perhaps as a rider.

The theorem that the perpendicular is the shortest distance scarcely needs proof, as it must have been treated in stage A; but if the opportunity is taken to prove it by the image method, it is an admirable point at which to introduce the image of a point in a line.

The light-path theorem, viz. that the route *ALB* is the shortest, follows at once by the use of the image. The result, which is by no means obvious, is of great intrinsic importance and interest [suggesting that light is reflected so as to take the shortest route]. The value of the group is enormously increased if this theorem is included. It leads to admirable riders, and in fact the only possible excuse for its omission is that Euclid omitted it.[1]

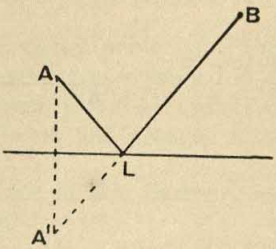

FIG. 61.

In addition to affording later on a convenient construction for the tangent to an ellipse (considered as a bifocal conic) this theorem leads up to a set of riders specially loved by small boys, viz. the constructions for striking the object billiard ball off one, two or three cushions of a perfect table. These in turn lead to the discussion as to the maximum number (5) of surfaces which a spider may have to crawl on, so as to reach a fly on the opposite wall of a room by the shortest route.

Here, if no more theorems are included in the group as 'standard' theorems, one comes to objection (ii) that the riders on this group are too hard.

This is emphatically not the case with those on the 'image' idea mentioned above, but it is true that boys when confronted with such a rider as 'Prove that the sum of the sides of a triangle is greater than the sum of its medians' are completely nonplussed.

The reply to this objection is that the art of getting boys to do these riders is (like the art of drama) the art of preparation. It is necessary to select and arrange so that one thing leads up to another, and especially it is nearly always necessary to give a hint as to the construction.

For example, a hint or two will enable the other Euclidean inequality theorems (best taken as riders) to be tackled.

1. To prove that as *P* moves round the circle from *B* to *B'* (Fig. 62), the length *AP* continually increases (which is the necessary

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[1] It is curious how different two names for the same thing can be. What is called above for obvious reasons the light-path theorem was dealt with in D. Mair's well-known *Exercises in Mathematics*, in a section of some length entitled 'On carrying water'.

basic fact for the crank and connecting rod mechanism), one must give the construction, 'Let AP' meet bisector of $\angle POP'$ in V, join VP'.

2. To prove that if P is inside the triangle ABC, then

$$BP+PC<BA+AC \quad \text{and} \quad \angle BPC>\angle BAC,$$

one must give the construction and hint, 'Let BP meet AC at V; compare BPC with BVC and then BVC with BAC'.

[These two cover the material of *Euclid* I, 21, 24, 25; III, 7, 8.]

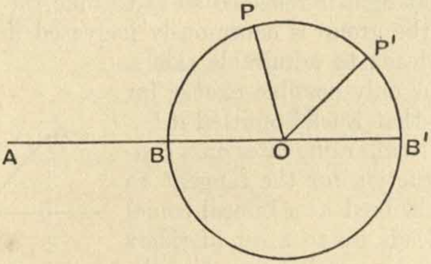


Fig. 62.

3. The median rider mentioned above may be associated with 'the sum of the sides of a quadrilateral is greater than the sum of the diagonals, but less than twice this sum', to give the idea of using the property that two sides of triangle are together greater than third. It may be set thus: 'By producing the median AD of $\triangle ABC$ to K, so that $DK=AD$, prove that $AB+AC>2AD$. Deduce that the sum of the three sides of the triangle is greater than the sum of its three medians.'

The third objection has a certain force; it is perhaps mainly in certain converse theorems that practice in handling these geometrical inequalities will prove useful. However in the consolidating stage if it is desired to point out the difference between deduction and proof, for not-too-weak pupils, this group of theorems may very conveniently be used thus:

The quite obvious result that two sides of a triangle are together greater than the third follows from the theorem that the greater angle is opposite the greater side. On the other hand, if this obvious result is taken for granted, a direct proof gives that greater side is opposite greater angle, and a proof by exhaustion gives the original theorem.

Thus deduction can be applied in more than one way according to the choice of fundamental assumptions, and it is not till this choice is made *and laid down as a basis* that the application of the deductive process becomes proof. It is doubtful if a better illustration of this is available in elementary geometry.

The Pentagon Group. This most elegant section of geometry is nowadays often omitted, very probably to the advantage of the weakest groups of certificate candidates, but certainly not without loss to the stronger pupils.

It consists mainly of constructions, the fundamental proposition being to divide a line in medial (or golden) section. This construction exactly follows and conveniently revises the solution of a quadratic equation by completing the square.

Next comes the construction of the 36°, 72°, 72° triangle, and the consequent possibility of constructing regular decagons and pentagons, the latter being specially interesting from their connection with the regular solids.

It is quite feasible to leave the discussion at this point, but if the most compact construction of the regular pentagon is desired it is necessary to include in the group one theorem, $p^2 = d^2 + r^2$, where p and d stand for the sides of regular pentagon and decagon in a circle of radius r.

It may be well to indicate Euclid's proof of this theorem, as *Euclid* XIII 10 may not be easily accessible.

If AB is the side of a regular pentagon and AK, KB are two sides of a regular decagon in a circle, centre O, and if M is the mid-point of AK, the angles in the figure are as marked where $\alpha = 18°$.

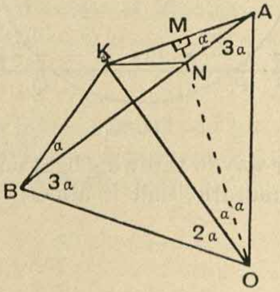


FIG. 63.

Then by similar triangles BAO, BON

$$BA \,.\, BN = BO^2,$$

and by similar triangles ABK, AKN

$$AB \,.\, AN = AK^2,$$

$$\therefore\ AB^2 = AB(AB + NB)$$

$$= AK^2 + BO^2,$$

$$\text{i.e. } p^2 = d^2 + r^2.$$

The 'most compact' construction to which it leads is this. Take perpendicular diameters AB, CD; bisect CO at P; let the circle centre

P, radius PA, cut PD at X. Then OX is the side of a regular decagon and AX the side of a regular pentagon inscribed in the circle.

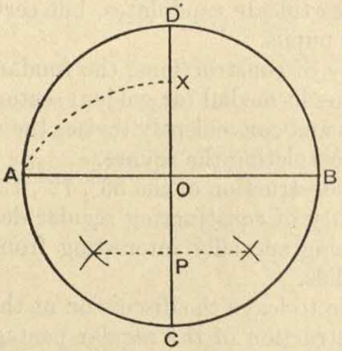


Fig. 64.

Further it is desirable with top divisions to include the more general solution of a quadratic equation by the circle shown below.

OP, OQ are roots of $x^2 - ax + bc = 0$.

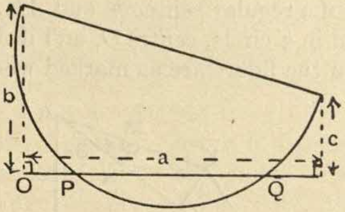


Fig. 65.

This provides another way of throwing light on the sum and product of the roots of a quadratic, and is almost worth while for that purpose alone,

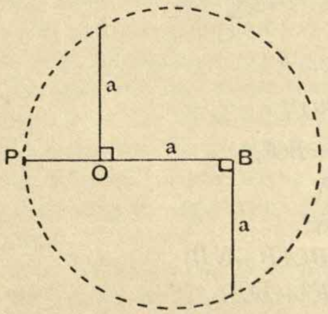


Fig. 66.

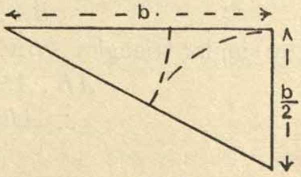


Fig. 67.

the comparison of the two figures 66, 67 for medial section being full of instruction.

As a final note it may be mentioned that if a long narrow strip of paper (say 1 foot long, 1 inch wide) is folded into a knot and the knot smoothed out flat without crumpling, a regular pentagon is formed at the knot. To demonstrate this by practice and theory will be found interesting.

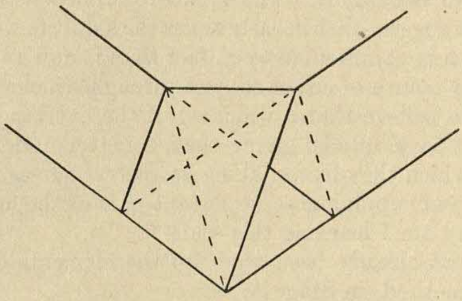


FIG. 68.

Medial Section and Logarithms. In Fig. 66, *OB* is the larger medial segment of *BP*.

If *OB* is unity let *BP* for convenience be called λ, then

$$\lambda(\lambda-1)=1 \quad \text{or} \quad \lambda^2=1+\lambda \quad \text{or} \quad \frac{1+\lambda}{\lambda}=\frac{\lambda}{1}.$$

This enables a series of points on the exponential or logarithmic curves to be constructed very easily with ruler and compasses by addition of ordinates, instead of repeated construction of third proportional.

At 0 and 1 erect ordinates 1 and λ; at 2 erect $1+\lambda$; at 3 erect $(1+\lambda)+\lambda$ and in general at each succeeding point erect an ordinate which is the *sum* of the two previous ones.

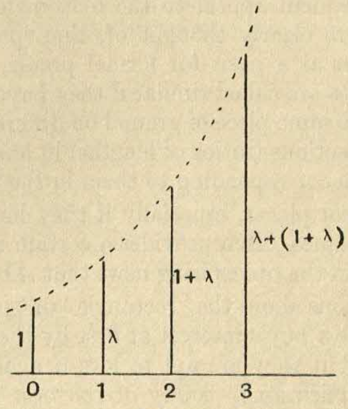


FIG. 69.

The tops of these ordinates lie on the curve $y=\lambda^x$ or $x=\log_\lambda y$ and a change of scale along the x-axis will convert the curve into the graph of $y=a^x$ for any desired value of a.

(This method was communicated by Mr. B. C. Wallis.)

Contacts with Geography. The kindred derivations of geometry and geography suggest that in early times the subjects were regarded as having much in common, more in fact than is now to be found in the elementary course of either subject through which boys are put at school. We believe that a widening of the overlap would be of real benefit to boys' minds, giving them a better understanding of the world in which they live, making geometry increasingly a practical subject, and supplying an important part of the answer to the question ' Why am I learning this stuff? '

Reference has already been made to the elements of surveying by graphical methods in Stage A.

At a more advanced stage of mathematical education than is dealt with in this report, the exigencies of examinations prevent mathematical boys from learning a great deal of mathematical geography which should interest them deeply, such as calculation of great circle courses, resection by calculation, tides and map-projections. While expressing a pious hope that these defects may be remedied in the future, we pass on here to the more elementary applications which are within our present scope. As these are separate suggestions, they are numbered, though not necessarily in order of teaching : some readers may be surprised to see how many chances for correlation they have neglected.

(1) Most boys are familiar with figures which are of the same shape before they learn the formal geometry of ' similar figures ' ; and this familiarity has been gained very largely by the use of maps. To many boys a frequent appeal to the map makes similar figures more real, and more clearly thought of, than purely geometrical words ; and, except as a basis for formal proofs, a good working definition is ' Figures are called similar if they have the same shape, like two maps of the same piece of ground on different scales, so that all angles and all fractions (ratios of lengths) in one are equal to the angles and fractions corresponding to them in the other '.

(2) The use of real places, especially if they have good Cornish, Welsh or Chinese names, often provides a certain zest which is conspicuously lacking in the uninspiring news that $AB=2{\cdot}4$ inches.

(3) The propositions about the ' rectangle ' or ' product ' property of the circle capture a boy's interest at first by their simple and unexpected results ; but they are apt to lose it again if not followed up by concrete applications. Many of the best of these are geographical, such as ' Given that the straight line joining two points

two miles apart on level water is eight inches under water in the middle, find the diameter of the Earth '. To calculate enormously great distances from measurements of a manageable size gives one a pleasing sensation of command: it begins to explain how astronomers, by measuring very small quantities, can talk with confidence of light-years, and at the same time introduces a boy by easy stages to the idea of orders of large and small quantities.

Given the heights and distances of three hills which appear on the map in a straight line, does the middle one cut off the view between the others? This kind of question, the Earth being taken as flat, is often set; but the interest and value of it are greatly increased by taking into account the curvature of the Earth; and if the data are well chosen the calculation is very simple, particularly if the metric system is used.

(4) Many boys, if asked the meaning of latitude, longitude or temperature, will answer cheerfully, 'Degrees' (possibly adding 'And shake the thermometer when you've taken it'), but with little or no idea as to which, if any, of these are angles. The meanings of latitude, longitude, elevation, depression and the nautical mile should be included in even the most elementary course of geometry; and by the time he reaches the school certificate stage the ordinary boy should be able to find the distances by various routes, which may be great circle, parallel of latitude or chord, between two places of given equal latitudes or longitudes (or unequal if his intelligence is above the average).

(5) In the absence of facilities for finer methods, the position of a point on the ground has often to be fixed on the map by resection, that is, by measuring the angles between the directions of three visible land-marks which are already mapped and proceeding by drawing or calculation. The solution by drawing gives excellent illustrations of the use of the locus-propositions associated with angles in a circle and of the distinction between good and bad intersections.

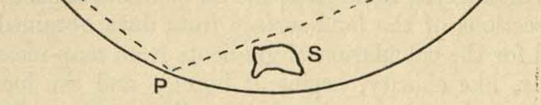


FIG. 70.

(6) Further illustrations are given by the horizontal and vertical 'danger angles' used in navigation. An arc is drawn in a horizontal plane through the positions of a lighthouse and a church to clear a shoal *S* by a safe distance. As long as the sextant angle between

L and *C* is not allowed to exceed $\angle LPC$, the horizontal danger angle, the ship will not run on to the shoal.

The vertical danger angle provides a simple exercise in three dimensions. It is desired to pass clear of a rock which is near a lighthouse. An arc is drawn with the lighthouse as centre to clear the rock by a safe distance : the angle subtended at a point on the arc by the light and the water-line at the base of the lighthouse is the vertical danger angle, and the sextant angle between these points must not exceed this. The height of the light given in the Light List is measured above high water : then if no allowance is made for tide the ship is still safe.

(7) The connection between the area and spherical excess of a spherical triangle needs no spherical, or even plane, trigonometry for its appreciation, and is best learnt by reference to a globe representing the Earth, with three coloured strings tied round it.

(8) The lines of greatest slope on a plane are often best found by drawing a horizontal line first and then a perpendicular to it. (Instances. A. At the corners of a 200-foot square on level ground vertical borings are made : they meet a plane sloping bed of coal at depths of 73, 94, 122 and 101 feet. Find the inclination of the coal-bed. B. Find by drawing the inclination of a sloping plane given in plan and elevation by its intersections with a given floor and wall.)

On curved surfaces such as that of hilly ground, as well as on planes, the lines of greatest slope and their horizontal projections cut the contours at right angles, a principle which deserves more attention than it commonly gets.

(9) There is great scope for the setting of questions of which the data are to be found in the atlas or the encyclopaedia, such as ‘ Trace a map of Arizona and measure its area ’ ; ‘ How much of Ireland would be visible from Snowdon if the cloud cleared off it ? ’ ‘ How much longer is the midday shadow of Peterborough Cathedral tower in winter than in summer? ’ ; or ‘ How high must the captain’s eye be for him to see the lights on the Caskets and St. Catherine’s Point at the same time? ’

(10) Whether we classify it as geometry or as a sub-section of graphs in arithmetic, some place should be found for the drawing of vertical sections of the land-surface from data obtained from the map, and for the calculation of gradients from map-measurements. Geography, like charity, begins at home ; and the local six-inch map gives good sections except in very flat country : later there is much interest in a longer and more varied section, such as that from Boulogne through Clapham Junction, the Chilterns, Snowdon and Lough Erne to Donegal. Continue it, and that vague and unconvincing phrase ‘ The Continental Shelf ’ assumes a sudden and startling reality.

(11) Eratosthenes (276 to 196 B.C.) observed that at midsummer the Sun shone straight down a well at Syene (Assouan), while at Alexandria on the same day the length of the shadow of an obelisk was $\frac{1}{8}$ of its height. From a rough measurement of the distance *AS*, which is approximately North and South, Eratosthenes deduced the circumference of the Earth as 252,000 stadia, about 25,100 nautical miles.

The finding of the angle is a good subject for exercises in geometrical drawing at first, and better still for trigonometry later.

(12) Problems may conveniently be set illustrating differences of local time. [E.g. V. le Neve Foster's *Geometry*, p. 49, q. 18. If it is 10 p.m., 31st July, at New York (73° 58′ 30″ W.), what is the time at Athens (23° 43′ 15″ E.)?]

(13) The Earth's angle of tilt should be considered, and its significance in fixing the Arctic and Antarctic Circles and the Tropics.

VIII STAGE C *a*

The Organisation of the Derived Propositions. In the 1923 Report the work of Stage C was discussed under two headings, 'the organisation of primitive propositions' and 'the organisation of derived propositions', and perhaps more emphasis was laid on the former. In this report we reverse the emphasis. We regard the organisation of primitive propositions as too difficult for most pupils, and have called it 'Stage C *b*', while we urge that the organisation of derived propositions, which we call 'Stage C *a*', should be undertaken by the great majority of our pupils and should form an important part of the work in the certificate year. If it is objected that this is too difficult it may be pointed out that because a boy is not easily able to bring to light and discuss the assumptions he has been unconsciously making, it by no means follows that he is incapable of organising knowledge which is familiar to him. Furthermore it is possible to show a boy that faith underlies his knowledge and yet to avoid a philosophical discussion of the contents of the faith itself.

Most boys can be taught to appreciate the fact that the whole of their geometrical knowledge is logically connected together. Without this consolidation the work of Stage B has a disconnected aspect; with it, we have at the lowest a means of revising without boredom the whole of Stage B geometry.

Logical Structure. While in Stage B we are concerned with obtaining, memorising and using geometrical facts, in Stage C *a* we take these same facts and weld them into a logical whole. But some of the logical structure of geometry will already have been brought to light; thus theorems will have been learnt in groups and in most of the groups one of the theorems will have been singled out as fundamental. This fundamental theorem is not necessarily the most useful or the most interesting one of the group, but it is the one a knowledge of which leads most directly to the proof of the others. Emphasis on this point of view helps the boy to understand the logical structure.

Again, the skeleton of logical structure shows through the flesh when we deal with converse properties. It is not suggested that the first occasion when converse properties arise is a suitable one for dealing fully with the logical relation of a theorem to its converse; but at some time rather late in the work of Stage B, the *proving* of converse properties will be dealt with in a short series of lessons specially set aside for the purpose. In the course of these lessons, two things relating to logical structure will be taught; that not all converse statements are true, and that if in proving Y we assume X, then in proving X we must not assume Y. Even if Stage C *a* is not attempted, a boy should not have followed a course of deductive geometry without having been made aware of the fact that circular arguments are fallacious.

Method for Stage C a. The first step in Stage C *a* will be a review of the grouping of the thoerems of geometry with special attention paid to the order in which it is possible to establish these theorems once the fundamental theorem is known. In passing, it may be noted that the group arrangement is not perfectly tidy. If, for example, the alternate segment theorem is included in the group concerning angles and circles, its proof needs in addition the fundamental property of the tangent to a circle.

After revision of groups of theorems in the manner suggested it is a short step to the process of tracing any theorem to its source in the properties of parallel lines, or of congruence, or both. This process forms the core of the work of Stage C *a*. The method will be illustrated by the following examples.

The theorem that the opposite angles of a cyclic quadrilateral are supplementary can be proved if we know that the angle subtended by an arc of a circle at the centre is twice the angle at the circumference standing on the same arc. We then ask what facts we must know if we are to prove this. Having obtained these facts we proceed in like manner and arrive at the following scheme:

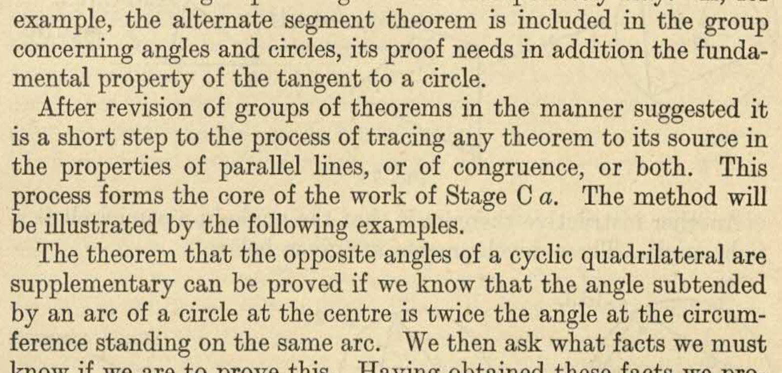


FIG. 71.

We then observe that this theorem can be proved in other ways, and although we get different developments they all come to the same end. In the following figures dotted lines show the constructions added, and if the proofs are followed out they will yield the schemes shown.

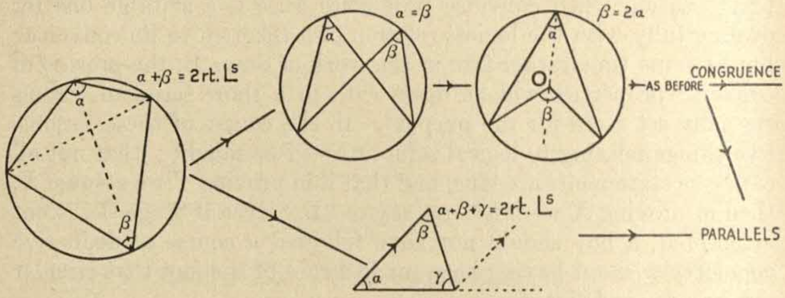


FIG. 72.

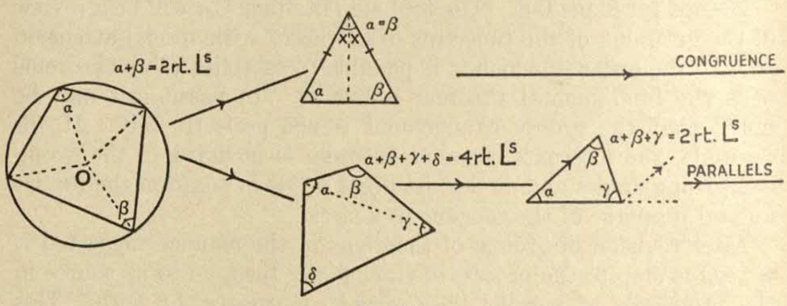
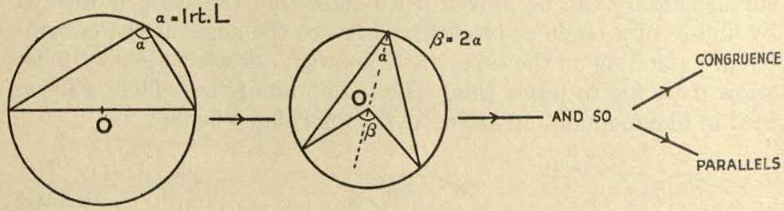


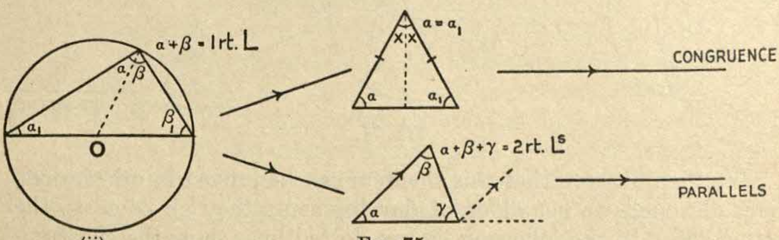
FIG. 73.

Another instructive theorem is that the angle in a semicircle is a right angle. Three developments are given below.



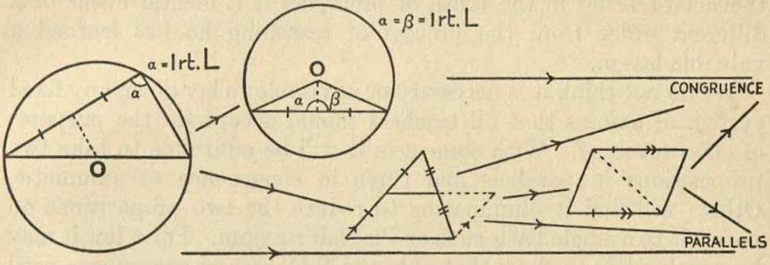
(i)

FIG. 74.



(ii)

FIG. 75.



(iii) FIG. 76.

In this way it is shown that the theorems which the boy has already learnt to prove depend fundamentally on the theorems about congruence and parallels.

There remains a second part of Stage C *a*. Certain theorems have previously been assumed as too obvious to require proof, and others have been accepted after discussions of the type used in Stage A. These include many converses and perhaps the theorems about similar triangles. They are now required as links in the logical chain, and they should be proved rigorously. It will be of interest to see how these proofs of familiar facts fit into place in a logical sequence, and to see that some of the facts previously assumed may be deduced from propositions still more fundamental. Without such a source of interest the proofs of many simple facts are apt to be wearisome.

If the treatment outlined above has been followed, the argument will have been of the form: 'We can prove *A* if we already know *B* is true; we can prove *B* if we already know *C* is true; and, continuing in this way, we can prove all our theorems in geometry if we already know the theorems about parallels and congruence to be true'. It will inevitably be asked whether these theorems themselves can be proved. And with this question we come, at the end of the course, to those considerations of the nature of geometry which in earlier days bulked so large at the beginning of the course. The answer to the question can easily be framed in general terms: 'We can only *prove* the properties of parallels and congruence if we first say we know something else to be true'. The average secondary school boy will assent to this without question. It now requires but little skill and time to generalise the idea that any conclusion from argument rests not only on the argument but also on some prior belief, and that when two people argue and fail to convince one another it is not generally because one can reason and the other cannot, but simply because they start from different beliefs in the first place. In short, it is as vital to examine the assumptions of the other fellow as it is to examine his reasonings. If a boy learns that the act of belief in the truth of premisses is a mental event of a different order from the process of reasoning he has learned a valuable lesson.

We do not think it is necessary or advisable to lay down any fixed system of axioms that all teachers should accept for the purposes of class teaching. With some sets it will be advisable to take two propositions in parallels and three in congruence as axiomatic. Others will find it illuminating to reduce the two propositions on parallels to a single basis such as Playfair's axiom. For a few it may be worth while to show that only the *SAS* case of congruence need be assumed. Where we stop is a matter of taste and judgment. The vital point is that the boy should see that, at some point or other, a stop must be made.

The work that has been outlined above should not be learned by the boy with a view to its reproduction in examinations. It may be described as work in the appreciation of the nature of geometry. It is work for class discussion. It is an experience that as many boys as possible should have. For those who wish to go further the discussion of Stage C *b* on p. 119 may be of interest and the Report of 1923 is a mine of provocative information.

Other Aspects of Revision. The revision of the theorems of geometry has been discussed. The revision of the constructions is a less complicated matter: all that will be necessary here is to go through the standard constructions again, filling in any gaps that may have been left and showing the boys the ways in which the constructions are set and the varying ways in which they are used in practical applications.

Riders will mostly have been discussed as attached to groups of theorems; and it will be worth while to run through in each group the riders most frequently recurring, those riders which the teacher when he opens a new geometry book, feels certain of finding in their accustomed places. But in the main, collections of miscellaneous riders such as may be obtained from examination papers will be used.

There is another way in which riders may be revised, and that consists in seeing how standard theorems of different groups may be brought to bear on the same figure so as to yield different properties of it. Thus the usual figure of a triangle cut in two by a line parallel to the base may be tackled by the area theorems to yield equivalent figures, is fundamental in the theory of similar figures, and suggests an important construction if the problem is to make the transversal cut off a given fraction of the area. Again, various methods may be used for dealing with the orthocentric quadrangle, as shown on p. 97. Such varied ways of looking at familiar figures are valuable in the time devoted to revision. A series of examples on the cyclic quadrilateral is given below.

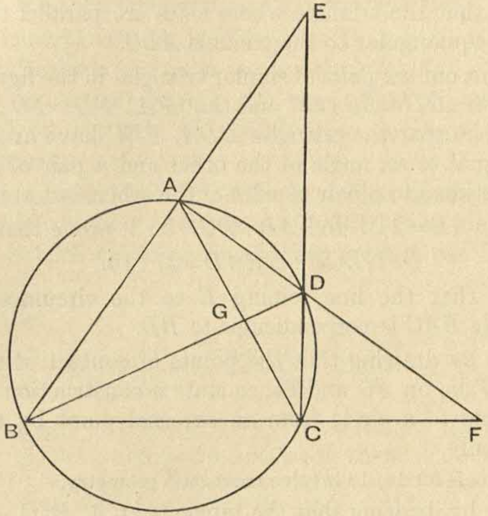


FIG. 77.

Given a cyclic quadrilateral $ABCD$ whose opposite sides meet when produced at E and F and whose diagonals meet at G.

(1) Prove without using the centre of the circle that

$$\angle BAD + \angle BCD = 2 \text{ right angles},$$

and state this result in a form involving the exterior angles.

(2) If BD bisects the angles at B, D, prove that $\angle BAD$ is a right angle.

(3) If $\angle BGA = 94°$, $\angle BEC = 28°$, $\angle AFB = 40°$, calculate all the angles in the figure.

(4) Prove that

$$\angle E + \angle F = \angle ADC - \angle ABC = 2 \text{ right angles} - 2\angle ABC$$

and that $$\angle E - \angle F = \angle BAD - \angle BCD.$$

Hence express E and F in terms of the four angles of $ABCD$.

(5) Show that G cannot be the centre of the circle.

(6) Prove that the angles at G are the sums and those at E and F the differences of the angles standing at the circumference on certain arcs shown in the figure.

(7) Prove that the bisectors of the angles at E and F are at right angles (see p. 139).

(8) If AD, DC, BC are equal and $\angle ACB = \theta$, find all the angles of the figure in terms of θ.

(9) Prove that $BG>GC$ and $AG>GD$ independently; also show that one of these facts follows from the other.

(10) Prove that the triangle whose sides are parallel to DA, DB, DC is equiangular to the triangle ABC.

(11) (i) Point out six pairs of similar triangles in the figure; deduce that $BG \cdot GD=AG \cdot GC$ and that $FA \cdot FD=FC \cdot FB$.

(ii) Prove that the triangles EDA, FDC have an angle of the one equal to an angle of the other and a pair of sides of one proportional to a pair of sides of the other, but are not similar.

(12) If $BC : AD=2 : 1$ and $AB : DC=5 : 3$, prove that

$$\triangle DCF : \triangle EAD=27 : 16.$$

(13) Prove that the line joining E to the circumcentre of the triangle EAC is perpendicular to BD.

(14) Verify by drawing that the points of contact of the tangents from F lie on EG and hence state a construction for drawing tangents to a circle from an external point by the use of a ruler only.

The proofs for 14, 15 involve cross-ratio geometry.

(15) Verify by drawing that the tangents at A, B, C, D each meet the next one either on EG or on GF.

(16) If AD, DC, BC are equal and $\angle BCA=2\angle ACD$, prove that BD, AC divide each other in medial section.

(17) Using the extensions of Pythagoras' theorem (or otherwise) prove that $EF^2=$ sum of squares of tangents from E and F to the circle [draw EX perp. to AF and mark off $XT=AX$ so that X bisects AT; or see No. 22].

(18) If A, B, C are fixed and D moves on the circumference from A to C what is the locus of the mid-point of EC and what the locus of the mid-point of CD?

(19) Prove that the angle between the tangents at A and C is $\angle ADC - \angle ABC$.

(20) Prove that $AB \cdot CD+AD \cdot BC=AC \cdot BD$ [take L on BD so that $\angle BAL=\angle GAD$; use two pairs of similar triangles thus made].

(21) If ABC is equilateral prove that $DA+DC=DB$, but if $AB=BC<AC$, prove that $DA+DC>DB$.

(22) Prove that the circumcircles of the triangles ABF, BCE, ADE, DCF are concurrent at a point K on EF. Use the point K to obtain an alternative proof for No. 17.

(23) If $AD=DC$ prove that $AB \cdot AC=BG^2+AG \cdot GC$.

(24) If BG is perpendicular to AC, prove that

$$AB \cdot AC=BG \cdot (\text{diam. of the circle } ABCD).$$

(25) Prove that the feet of the perpendiculars from D to AB, BC, CA, are collinear and that the line of collinearity bisects the join of D to the orthocentre of the triangle ABC.

(26) If AC is perpendicular to BD, prove that BC is twice the perpendicular from the centre of the circle to AD.

(27) If AC is perpendicular to BD, prove that the perpendicular from G to any side of $ABCD$ when produced bisects the opposite side.

(28) If AC is perpendicular to BD, prove that the feet of the perpendiculars from G to the sides of $ABCD$ form a quadrilateral which is both cyclic and such that a circle can be inscribed in it.

(29) Prove that

$$\frac{EA}{DE} \div \frac{AB}{CD} = \frac{(\text{Tangent from } F)^2}{BF^2}.$$

(30) With the aid of trigonometry, prove that if $AB=a$, $BC=b$, $CD=c$, $AD=d$, and $2s=a+b+c+d$, then

$$\text{area } ABCD=\sqrt{(s-a)(s-b)(s-c)(s-d)}\,;$$

$$AC^2=\frac{(ac+bd)(ad+bc)}{ab+cd}, \quad BD^2=\frac{(ac+bd)(ab+cd)}{ad+bc};$$

$$EF^2=(ad+bc)(ab+cd)\left\{\frac{bd}{(b^2-d^2)^2}+\frac{ac}{(a^2-c^2)^2}\right\};$$

$$\text{area } EFG : \text{area } ABCD=2abcd : (a^2-c^2)(b^2-d^2).$$

Post Certificate Work for Non-Specialists. In this section some suggestions are made as to suitable topics in geometry, other than Stage C *b* work (see p. 119), for boys who have passed the school certificate stage but do not intend to work through a specific higher certificate course. The topics mentioned can all be developed from the ordinary elementary course of geometry, and can be made very interesting to quite unmathematical boys.

Elementary Geometry of the Conic Sections. The focus and directrix property of a plane section of a right circular cone is easily obtained and gives an excellent exercise in solid geometry. (A model of a cone cut in two parts and showing the elliptic section is a useful accessory here.) From this property a few conics might be drawn on squared paper, and the angle subtended at the focus by the part of a tangent between its point of contact and the directrix investigated first by Stage A and then by Stage B methods. When this is done a large number of easy properties follow, among them the principle of a parabolic reflector.

Inversion. This is an interesting method of transforming one figure into another, and deducing properties of the first from known (or easily proved) properties of the second. Stage A methods can again be used with advantage at first. In most applications there is little reference to the lengths of lines, but an example to the contrary is Ptolemy's theorem about the sides and diagonals of a cyclic quadrilateral which follows from the fact that if ABC is a straight line then $AC=AB+BC$. The properties of orthogonal circles come into the work, which may be continued with a good set as far as a proof of Feuerbach's theorem.

Radical Axis and Coaxal Circles. This work provides some good drawing exercises, e.g. constructing a circle of a given system to pass through a given point. A drawing of a set of intersecting coaxal circles together with the associated coaxal system which has the common points of the first set as limit points, shows the plane divided into a network of curvilinear rectangles. This work provides interesting examples for inversion.

Orthogonal Projection of the Circle. If a circle is drawn on squared paper and all ordinates perpendicular to a given diameter are then reduced in a fixed ratio, the orthogonal projection is obtained. It is possible to verify by Stage A or Stage B methods that the curve is an ellipse as obtained from the focus-directrix property. Simple properties of conjugate diameters and the auxiliary circle, are easily obtained, also an expression for the area of an ellipse, and of an enveloping parallelogram.

More Advanced Plan and Elevation Drawing. For example, the elevation of a right pyramid on a square base on a vertical plane making an angle of 30° with a side of the base ; the section of the solid by a parallel plane ; the section of a right circular cone by a plane parallel to the axis ; the section of a circular cylinder or hexagonal prism which lies on a horizontal plane by a vertical plane which is not at right angles to the length of the solid.

Other Accurate Drawing. For example, circles to satisfy given conditions of passing through points and touching lines or circles ; perspective drawing.

More Difficult Plane Geometry of the Triangle. The nine-points circle (see Appendix 7) ; Simson's line ; collinearity of orthocentre, nine-points centre, centroid, circumcentre ; isogonal lines ; Brocard points ; etc.

Map Projections.

Topics suggested as possible in Stage A. Envelopes ; curvature; etc. (see p. 14).

IX STAGE C *b*

General Discussion. It is assumed throughout this report that up to the School Certificate three cases of the congruence of triangles and the angle properties of parallel lines will be taken for granted.

It may be questioned whether even after this stage it is profitable for the majority of boys to embark upon the examination of those basic propositions rather than to go on and learn some fresh geometry. But it is probable that, as things are, some teachers will feel it incumbent upon them to follow the former course; it is quite possible that some boys may wish to do so and may even themselves raise questions about the logical validity of those propositions and their mutual relations. We therefore give some suggestions without wishing to imply that there is any obligation on teachers to deal with the matter at all.

The first step naturally is to show that each group can be reduced to a single assumption. As regards parallels this is obvious; if, for instance, we assume the equality of a pair of corresponding angles, the other two properties follow at once. As regards congruence the matter is not quite so simple. It has to be shown that if, in accordance with modern practice, we assume the case *SAS* (*Euclid* I 4) the other two can be logically deduced from it and that without use of parallels. The second case (*SSS*, *Euclid* I 8) can be proved, as in fact it generally is proved in current school books, by use of the angle property of isosceles triangles; but Euclid's proof by aid of his I 7 is well worth study.

It is the case of two angles and a side (*Euclid* I 26), which introduces new matter. For our school practice up to this point, assuming the angle properties of parallels, has made the angle-sum of a triangle familiar, and the essential point now is to prove the proposition without use of the angle-sum. Even if we disregard the familiar reprobation of superposition we can only prove half the proposition (*ASA*) by that means, and for the other half (*SAA*) we need a new method. The obvious course is to take Euclid's proof, at least for *SAA*. This means

(i) proving that the exterior angle of a triangle is greater than the interior opposite angle from Fig. 78 (*Euclid* I 16),

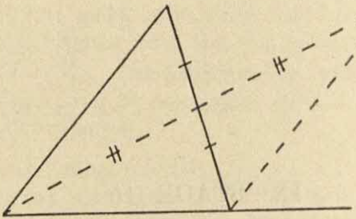


FIG. 78.

(ii) proving the *SAA* case of congruence by reduction to absurdity from some such figure as Fig. 79 (*Euclid* I 26).

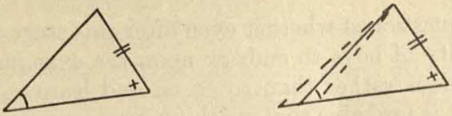


FIG. 79.

This is valuable in itself, for it brings to light the importance of the proposition that the exterior angle of a triangle is greater than either of the interior and opposite angles, and the fact that this property can be proved by congruence alone. Hitherto the boy will have regarded it as a mere corollary of 'exterior angle C=sum of interior angles A and B'.

This proposition (I 16) invites further consideration because its proof depends on the axiom that two straight lines cannot meet twice; therefore the proposition and its consequence (*SAA*) have only a limited validity on the sphere, whereas the other cases of congruence there hold completely; e.g. in a spherical triangle made by two meridians and a piece of the equator, an exterior angle at the equator is equal to one of the opposite interior angles and may be greater than, equal to or less than the other; and two such triangles need not be congruent.

The next stage, though far more interesting, is matter perhaps for specialists only. It naturally begins with the question whether the properties of parallels can be deduced from congruence, but it may also be asked, can congruence be deduced from parallelism? The answer in each case is 'no'; and this at least should be known, if only as dogma, by all.

We will deal with the latter question first. But it needs stating more fully, for it would hardly be supposed that from angle properties alone the metric properties of lines could be deduced. The question then relates rather to parallel translation than merely to the angle properties of parallels. Put shortly the question is this: if we assume that the opposite sides of a parallelogram are equal, can we deduce the general properties of congruence? We cannot, for though we can compare lengths along one line and its parallels and also along any other line and its parallels, we cannot compare the one with the other without making some further provision, in effect Euclid's postulate as to circles: all figures can be deformed indefinitely provided parallels remain parallel; there is no distinction, for instance, between a right angle and other angles, nor yet between an ellipse and a circle.

In considering whether properties of parallels can be deduced from congruence, pupils should first see clearly that, parallel lines being defined as lines which do not meet, then, the equality of corresponding angles (or an equivalent condition) being given, we can prove the lines parallel. To do this we use the theorem that an exterior angle of a triangle is greater than an interior and opposite one. We cannot however prove the converse theorem that if lines do not meet then the corresponding angles are equal. The position is then that, given a line and a point P outside it, we can draw a line parallel to the given line by making equal corresponding angles on a transversal through P, but we have no means of deciding whether different transversals will give different parallels through P or the same one.

An assumption has to be made; and the different assumptions or different forms of the same assumption can be compared and contrasted.

Some of the best known of these assumptions are as follows:

(i) Euclid's.

That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.

This seems far more clumsy and less self-evident than some of the possible alternatives, and at one time mathematicians were apt to speak of it as a vicious assumption, a blot on Euclid's system.

The more modern view, however, is that the stark arresting ugliness of Euclid's assumption is an outstanding merit, a triumph of genius comparable with his treatment of ratios. For it is essential to stress the point that an assumption is necessary, and the more this is disguised by a specious appearance of inevitable truth the greater the danger that the necessity of making an assumption will be overlooked.

This danger is perhaps greatest in the case of the next assumption.

(ii) Playfair's Axiom.

Two intersecting lines cannot both be parallel to the same straight line.

The choice between (i) and (ii) makes no difference to the order of proof, which is the traditional one.

In either case parallel lines are defined as lines in the same plane which do not meet when produced, and it is proved first that if alternate angles are equal the lines are parallel (since if they met, the exterior angle of a triangle would be equal to an interior opposite angle).

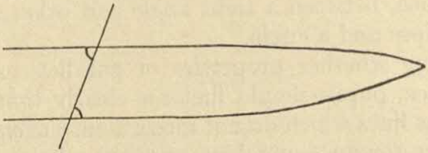


FIG. 80.

In each case the assumption is used in proving the converse theorem that if the lines are parallel, then the alternate angles are equal.

With Euclid's postulate, if, in Fig. 81,

$$a>b, \quad b+c<a+c, \text{ i.e.} <2 \text{ right angles},$$

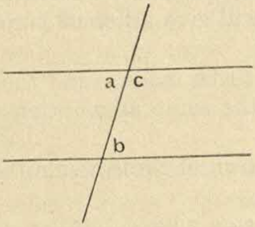


FIG. 81.

so that the lines would meet contrary to hypothesis.

With Playfair's axiom, if, in Fig. 82, a is greater than b, draw RT so that $\angle PRT=a$, then both RS and RT are parallel to PQ contrary to the axiom.

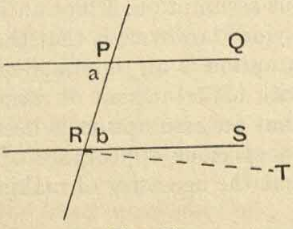


FIG. 82.

(iii) The principle of similarity.

This was discussed at some length in the 1923 Report, p. 35. No text-book has yet (1938) been published in which this idea is worked out in detail. Sir Percy Nunn, however, has recently written an account of the way this can be done.

It will be published in the *Mathematical Gazette* in the course of 1938 and a brief extract from it will be found in this report on p. 130.

(iv) At least one rectangle, that is a four-sided figure whose angles are all right angles, exists. For a discussion of this, see p. 125.

Teachers who deal with Stage C *b* should certainly be aware that there are these and other possibilities of replacing the 'postulate of parallels' by some apparently different assumption, but from the teaching point of view the traditional order based on Euclid's postulate or Playfair's has the merit that the proofs are sufficiently easy not to distract the pupils' attention from what is at this stage the main issue, namely the study of the logical structure of geometry.

In considering this logical structure it is important to establish that a certain group of facts, associated with parallels, similarity and the angle-sum of a triangle, hang together. We cannot have any one of these facts without also having all the others, and if we discard one, the whole group disappears. This is shown logically by assuming one fact and proving the others from it, as when, for instance, we assume Playfair's axiom and prove the whole group of theorems in the traditional way.

When we find that we cannot prove the assumed fact from theorems outside the group, the question arises whether the assumed fact is necessarily true, and we are then faced with the possibility of a geometry in which the whole group is non-existent.

It is always difficult to realise that a proposition is an assumption and not a certain truth unless we can conceive the possibility of its being untrue. If we want to bring home the fact that the parallel postulate, or whatever alternative assumption we use, is not a necessary truth, and at the same time to illustrate the dependence on it of the similarity theorems and the angle-sum property of the triangle, it is well worth while to discuss the geometry of great circles on a sphere.

Such great circles share many of the properties of straight lines in a plane, and in fact if we confine ourselves to a sufficiently small part of the surface they are indistinguishable from straight lines in a plane, as we know from our experience of the surface of the earth. If, however, we consider a larger part of the surface, we find that only certain of these properties hold good. It is still possible to construct in a given position a spherical triangle congruent to a given spherical triangle (or if we prefer to put it so, a spherical triangle can be moved about freely on the surface); so the geometry based on congruence will still be valid. But if we produce two arcs of great circles which make equal corresponding angles with another great circle (e.g. two meridians, both at right angles to the equator) it is clear that they meet one another. In fact any two great circles meet, and so parallel great circles do not exist. From the same figure we see that the angle-sum property does not hold good for spherical triangles, and it is also quite easy to see that if we attempt to draw a triangle equiangular, and so similar, to one formed by the equator and two meridians, but on a larger arc of the equator as base, our two great circles drawn at right angles to the base still meet at the pole, and the attempt at similarity has therefore failed.

These considerations are sufficient to illustrate the interdependence of similarity, parallels and the angle-sum property of the triangle, but we may also, if we wish, go one step further and discuss non-Euclidean geometry. Having seen that the parallel postulate is an assumption, and that without it or some equivalent we cannot develop Euclidean geometry, it is natural to enquire into the consequences of rejecting the assumption. We then have to consider the possible alternatives. Consider a straight line *AP* drawn through a fixed point *A* to meet a fixed straight line at *X* (Fig. 83). If *AP* rotate about *A*, *X* will move along the line. In Euclidean geometry we suppose that there is no limit to the distance which *X* moves before *AP* reaches a particular position *AL* in which it does not meet the fixed line at all and which we describe as 'parallel' to the fixed line; and further, that if *AP* continue to rotate, it will immediately cut the fixed line again, the point of intersection *X* returning eventually to its starting point. There are two other possible suppositions, however, one of which is that after *X* disappears it does not reappear immediately. In this case the line *AP* rotates through an angle *LAL'* before it again meets the fixed line. The geometry based on this assumption is called 'hyperbolic'.

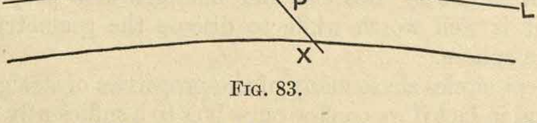


FIG. 83.

The other possibility is that *X* never disappears at all, and this can only happen if the fixed line, instead of extending infinitely in both directions, is 'closed', like the equator of the earth. Then *X* can return to its starting-point without ever disappearing. The geometry based on this assumption is called 'elliptic'.

We thus have three possibilities for plane geometry, of which the third leads to results very like those obtained for the geometry of great circles on a sphere. The non-existence, on a sphere, of parallels, of the angle-sum property and of similar figures is a consequence of the curvature of the surface, and it will be of interest to some boys to learn that when people talk about the ‘ curvature of space ’, they mean simply that the geometry of our space is peculiar in somewhat the same way as the geometry of the surface of the earth is peculiar to anyone who regards it as flat.

Those who wish to follow up non-Euclidean geometry in detail should study Lobatschewsky's *Theory of Parallels*, published in English by the Open Court Co.

Those who wish for a logical system of Euclidean geometry may refer to H. G. Forder's *Foundations of Euclidean Geometry*.

The Necessity for an Additional Axiom, that of Parallels or an Alternative. After his first 26 propositions, that is before beginning parallels, Euclid might have proceeded as follows :

Construct a quadrilateral $OAPM$ in which O, A and M are right angles. With only the axioms I have already been using, I cannot prove anything about its fourth angle P or the length of PM ; but I proceed to examine all the possibilities.

There are three of these : PM may be equal to, greater than or less than AO.

A. I maintain that if $PM=AO$, then also $PA=MO$ and P is a right angle ; that if $PM>AO$, then also $PA>MO$ and P is acute ; that if $PM<AO$, then also $PA<MO$ and P is obtuse.

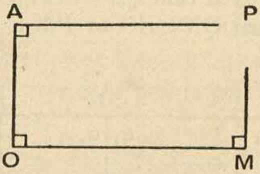


FIG. 84.

And conversely according as P is right, acute or obtuse, PM and PA are equal to, greater than or less than the sides opposite them.

B. I maintain further that whichever of these three possibilities is true for the original figure is true also for all such trirectangular quadrilaterals (*TRQ* for short).

C. And further that if $PM=AO$ the angle-sum of all triangles is two right angles ; that if $PM>AO$ the defect of the angle-sum from two right angles is proportional to the area ; and that if $PM<AO$ the excess of the angle-sum above two right angles is proportional to the area.

Proofs.

A. Suppose that $PM>AO$. Draw CH, the mid-point-perpendicular of OM, meeting AP at H. From MP cut off $MB=OA$. Then the quadrilaterals $HCOA$ and $HCMB$ are congruent.

$\therefore \angle HBM=A=$ a right angle.

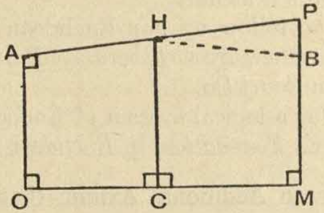


FIG. 85.

$\therefore$ P, the interior angle of the triangle HBP, is acute. Similarly if $PM<AO$, P is obtuse,; and if $PM=AO$, P is a right angle. Conversely, if P is acute, on making the same construction P must fall outside MB, or else the exterior angle of the triangle HBP would be less than the interior angle; and similarly for the other two suppositions.

B. If $PM=AO$, and $\therefore$ by A the figure is a rectangle, it is to be proved that rectangles also exist of all dimensions.

Let Q be any point in AP. Draw QN perpendicular to OM. Then it can be proved that $QN=AO=PM$.

For suppose that $QN<AO$ or PM: then $q_1>a$ and $q_2>p$ (by prop. A).

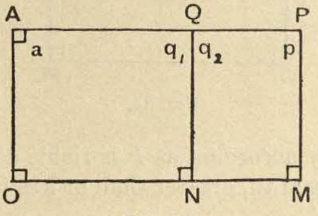


FIG. 86.

$\therefore$ $q_1+q_2>a+p$, which $=2$ right angles; and this is impossible. Similarly $QN \not> AO$.

$\therefore$ $QN=AO=PM$, and $q_1=q_2=$ a right angle.

Similarly common perpendiculars can be drawn to AO and PM, all equal to AP and OM.

Thus rectangles exist of all dimensions not greater than OM and OA; but since these rectangles can be repeated indefinitely and fitted together, rectangles of all sizes exist.

C. It follows in the usual way by congruence that the angle-sum of a triangle is two right angles.

Before proceeding to the other two cases of B and C, I prove the following proposition.

D. If from three points in one straight line, the perpendiculars to another straight line are equal, then all such perpendiculars are equal, and are perpendicular to both lines.

Let $PM=QN=RL$ (Fig. 87).

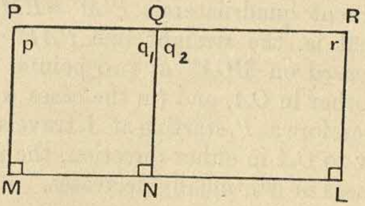


FIG. 87.

Then angle $p=q_1$, as proved in A,

and $r=q_2$,

and $p=r$,

$\therefore q_1=q_2$.

But $q_1+q_2=2$ right angles,

$p=q_1=q_2=r=$a right angle.

$\therefore$ by the part of proposition B, already proved, all perpendiculars from intermediate points are equal and are perpendicular to both lines.

So far we have proved that the Euclidean system follows from either of the two assumptions:

(1) that the fourth angle of a *TRQ* is a right angle,

or (2) that from three points in one straight line equal perpendiculars can be drawn to another.

Let equal perpendiculars be drawn in the same sense at all points of a straight line. Their free ends form a line (curved or straight) which we will call an equidistance locus (*EQD* for short) based on that straight line. We have just seen (prop. D) that if three of its points lie on a straight line, all do. Hence in the other two cases, in which the *EQD* is not straight, no straight line can cut it more than twice.

Now in the original figure of a *TRQ* produce *PA* and *MO* (Fig. 88), take $OM'=OM$ and erect the perpendicular $M'P'$ to meet *PA* produced at P'.

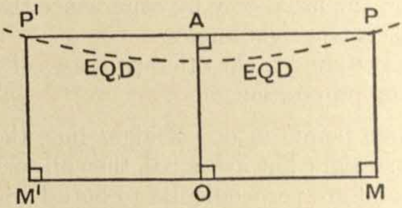


FIG. 88.

Then by congruent quadrilaterals $P'M'=PM$, $AP'=AP$, and angle $P'=P$: that is, the straight line PAP' cuts each of the successive *EQD* based on MOM' at two points, which are the reflections of each other in *OA*, and (in the cases we are considering) nowhere else. Therefore as *P*, starting at *A*, travels along the straight line perpendicular to *OA* in either direction, the perpendicular *PM* continually increases or continually decreases.

B_2. If then $PM>AO$ in the original *TRQ*, it is also greater than *OA* all along the straight line PAP'; and therefore again the angle *P* is always acute.

But since *P* is acute, $PA<MO$; and therefore from all points along the straight line *MP* produced either way the perpendicular to *OA* is greater than *MO*.

Combining the two results we see that in all *TRQ*, on this particular hypothesis, the fourth angle is acute.

B_3. Similarly, if originally $PM<AO$, the fourth angle in all *TRQ* is obtuse.

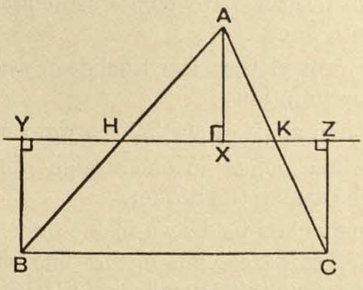


FIG. 89.

C_2. Let *ABC* be any triangle (Fig. 89). Bisect *AB* and *AC* at *H* and *K*. Join *HK* and draw *AX*, *BY* and *CZ* perpendicular to it.

Then by congruent triangles $BY=AX=CZ$, the area of the triangle = that of the quadrilateral $BYZC$, and the angle-sum of the triangle = the sum of the angles B and C of the quadrilateral.

If A moves along an equidistance locus based on YZ, AX remaining constant, H and K remain on the straight line YZ and the quadrilateral is unaltered. Therefore the area $\varDelta$ and the angle-sum $\varSigma$ of the triangle are also unchanged.

From this we can derive a construction for transforming triangles without altering their area or angle-sum: given two triangles, we can reduce them both to right-angled triangles with one pair of equal sides as in the figure, and so put them together to form a single triangle in which

$$\varDelta=\varDelta_1+\varDelta_2 \qquad (1)$$

and

$$\varSigma=\varSigma_1+\varSigma_2-\varPi \qquad (2)$$

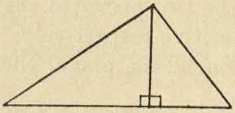


FIG. 90.

These equations hold for all our three original suppositions.

(1) If in the original figure $PM=AO$, B and C are right angles, and $\varSigma=\varPi$ for all triangles. The equidistance locus is a straight line.

(2) If $PM>AO$, B and C are acute, $\varSigma<\varPi$ and it is convenient to write D for $\varPi-\varSigma$. Then equation (2) becomes $D=D_1+D_2$.

(3) If $PM<AO$, B and C are obtuse, $\varSigma>\varPi$ we write E for $\varSigma-\varPi$ and $E=E_1+E_2$.

In cases (2) and (3), since the equations

$$\varDelta=\varDelta_1+\varDelta_2,$$

and

$$D=D_1+D_2,$$

or

$$E=E_1+E_2$$

can be extended to any number of triangles or to multiples of the same triangle, it follows that $\varDelta \propto D$ or $\varDelta \propto E$.

We have now examined all the possibilities and have shown that if we assume the ordinary Euclidean axioms, including that of congruence but not that of parallels, there are three possibilities: Euclid's system in which rectangles exist and the angle-sum of a triangle is constant, and two others in which the defect or excess of the angle-sum of a triangle is proportional to its area.

The third of our cases ($PM<AO$, $\varSigma>\varPi$) if pushed further leads to a conclusion which for some time was supposed to put it out of court. As the perpendicular diminishes it will presently vanish, i.e. the straight lines *MOM'* and *PAP'* will meet twice, once on each side of *AO*; and such a possibility was excluded by the axiom that two straight lines cannot meet twice. Consequently there remained two systems, Euclid's and Lobatschewsky's, which were supposed to be in competition with one another, one or other presumably being absolute truth.

In reality there is no such mystery about the matter. Euclid professed to be dealing with plane geometry—but without any working definition of a plane. Later the missing definition was supplied—but it depends on the straight line, which remained, and must remain, undefined. The subject he was treating up to I 26 was therefore not the geometry of a plane, but that of a two-dimensional continuum of such a character as to make congruence possible.

Of such continua or surfaces there are three types corresponding to our three suppositions, viz. those in which the Gaussian curvature, that is the product of the two principal curvatures, constant in all, is zero, negative and positive respectively. The simplest examples of the three are the Euclidean plane, the pseudosphere and the sphere.

To separate the first of these from the others requires an additional restriction which may take any one of many forms—Euclid's axiom of parallels; Playfair's modification of it; the existence of a single rectangle; the existence of three equal perpendiculars to a straight line from points in another; the existence of any pair of equiangular triangles different in size.

For educational purposes perhaps it is undesirable that any one of these or others (unless indeed for historical reasons it be Euclid's) should be adopted as the standard axiom. It may be better, having once made clear the central fact that a further assumption is necessary and why, to take several in turn and to show that they all amount to the same thing.

The Replacing of Euclid's Parallel Postulate by the Principle of Similarity. (A short extract from an article with this title by Sir Percy Nunn.)

> The principles of congruence and similarity were discussed in the 1923 Report: they are not needed in their full generality, in which they are assumptions about any figure.

We take as our axioms the following more restricted statements.

(I) Let *ABC* be any triangle and *DE* any straight line. Then upon *DE*, on either side of it, it is possible to construct a triangle *DEF* having the angles at *D*, *E* and *F* equal respectively to the angles at *A*, *B* and *C*.

(II) Given a triangle *ABC* and a line *DE* equal to *AB*, it is possible to construct on DE, on either side of it, a triangle DEF congruent with ABC.

Using axioms (I) and (II), the following sequence of theorems is suggested.

Theorem I. Let ABC and DEF be any two triangles in which two angles of the one are equal respectively to two angles of the other. Then the third angles must also be equal.

For in the triangles ABC and DEF, let the angles at A and C be equal to the angles at D and F. By Axiom I we can construct on DF, on the same side of it as E, a triangle equiangular with triangle ABC and having the angles at D and F equal to the angles at A and C. It follows that the sides of this triangle must lie along DE and FE and must therefore cut in E. Thus DEF is identical with the triangle on DF equiangular with ABC. Hence the angle at E must be equal to the angle at B.

Theorems II, III and IV. These theorems establish the three sets of conditions for congruent triangles. Theorem II will run as follows:

Let ABC and DEF be two triangles in which AB is equal to DE, the angle at A equal to the angle at D, and one of the other angles in ABC equal to the corresponding angle in DEF: then the triangles must be congruent.

By Theorem I the triangles are equiangular, so that the angles at A and B are equal to the angles at D and E. By Axiom II it is possible to construct on DE, on the same side of it as F, a triangle congruent with ABC, having the angles at D and E equal to the angles at A and B. It follows that the sides drawn from D and E will fall along DF and EF and must therefore cut at F. Hence the triangle DEF is identical with a triangle congruent with ABC and is itself therefore also congruent with ABC.

Theorems III and IV establish the other two main cases of congruence, the theorem of the isosceles triangle coming between them.

The next set of theorems develops the theory of parallel lines and transversals.

Theorem V. Let a transversal ABC (Fig. 91) *fall across two lines BL and CM in such a way as to make the corresponding angles at B and C equal. Then BL and CM cannot meet.*

Suppose them to meet in a point N. Let A be any point on the transversal such that B is between A and C. Join AN.

Then in the triangles ABN and ACN, the angle at A is common and the angle at B is equal to the angle at C. Hence by Theorem I the angle ANB is equal to the angle ANC, which is impossible. Hence the lines cannot meet.

Lines which lie in the same plane and do not meet are said to be *parallel.*

Theorem VI. If a transversal to a pair of lines makes the corresponding angles equal, then any other transversal to the same two lines also makes the corresponding angles equal. This important theorem has three cases.

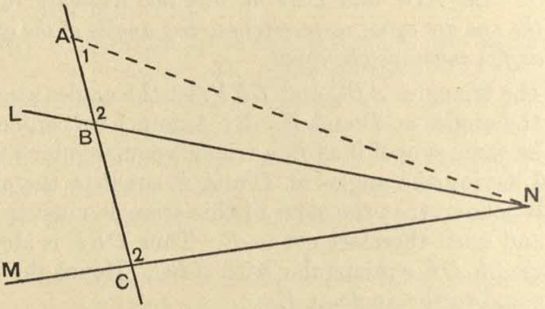


FIG. 91.

Case I. Let BD and CE be the lines (Fig. 92). Let the transversal ABC make the corresponding angles at B and C equal, and let the second transversal XDE cut the former in some point A which is not on either of the lines BD or CE.

Then in the triangles ABD and ACE, the angle A is common and the angle at B=the angle at C. It follows that the corresponding angles at D and E are equal.

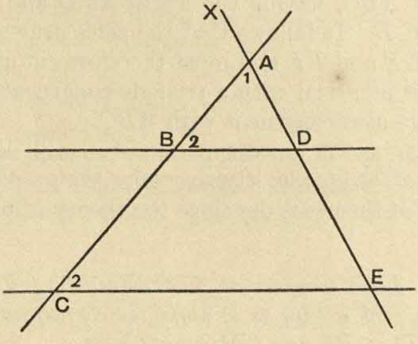


FIG. 92.

Theorem VI has also the two other cases, when the two transversals (i) meet on one of the lines, (ii) are parallel.

It is easy to show that equality of the corresponding angles made by a transversal cutting a pair of lines implies equality of the alternate angles, and conversely, etc.

Theorem VII. Let ABC be any triangle: then the sum of its angles must be two right angles. At A (Fig. 93) draw a ray AX, making the angle XAB equal to the angle ABC, and produce XA beyond A to Y. Then by Theorem VI, since the alternate angles XAB and ABC are equal, the alternate YAC and ACB are also equal. Hence the three angles of the triangle are equal to the three angles at A: therefore their sum is equal to two right angles.

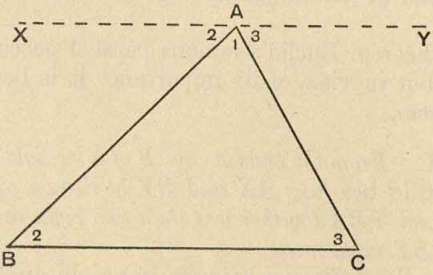


FIG. 93.

Theorem VIII. Conversely, it is always possible to construct a triangle whose angles are equal to three given angles, if the sum of these is two right angles.

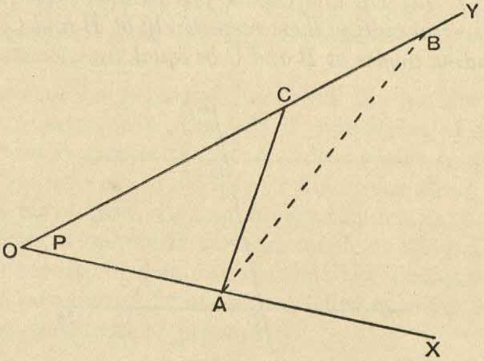


FIG. 94.

Let the sum of the angles P, Q and R be two right angles. From any point O (Fig. 94) draw two rays, OX and OY, making the angle XOY equal to the angle P. On OX take any point A and on OY any point B, and join AB. Then if the angle at A is equal to one of the two angles Q or R, the angle at B must be equal to the other of these angles, and the triangle required has been constructed.

If not, then one of the two angles at A and B must be greater than at least one of the two angles Q and R. Suppose that the angle at A is greater than the angle Q. From A draw AC, making the angle OAC equal to Q. Then AC must cut the ray OY in some point C between O and B. Also, since the angles at O and A are equal to the angles P and Q, the remaining angle at C must be equal to the angle R.

From this theorem Euclid's famous parallel postulate follows as a corollary; but in view of its importance it is best treated as a separate theorem.

Theorem IX. From the ends A and B of a straight line, and on the same side of it, let two rays AX and BY be drawn so as to make the angles XAB and YBA together less than two right angles. Then the rays AX and BY must meet.

For let C be the difference between two right angles and the sum of the angles A and B. Then by Theorem VIII it is possible to draw a triangle whose angles are respectively equal to A, B and C. Also by Axiom I it is possible to draw on AB, on the same side of it as the rays, a triangle equiangular with this. In one of its two possible positions the sides of that triangle drawn from A and B would fall along AX and BY. Hence AX and BY must intersect in the apex of the triangle.

Theorem X. Let BL and CM be two parallel lines (Fig. 95), *and ABC a transversal cutting them respectively at B and C. Then shall the corresponding angles at B and C be equal.*

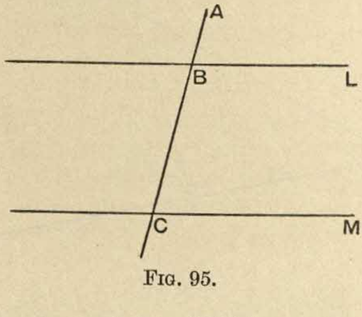


FIG. 95.

For if they are not equal, let ABL be greater than BCM. To both add the angle LBC. Then since the sum of LBC and ABL is two right angles, the sum of LBC and BCN must be less than two right angles. Hence by Theorem IX BL and CM must meet towards L and M. But they cannot meet, being parallel. Similarly if ABL is less than BCM the lines must meet in the opposite direction, which is again impossible. Hence the corresponding angles at B and C must be equal.

Theorem XI. As the last theorem of this series one can prove the parallel axiom in the form associated with the name of Playfair.

Let AL (Fig. 96) *be any line and P any point outside it. Then through P can be drawn one line and one line only parallel to the line AL.*

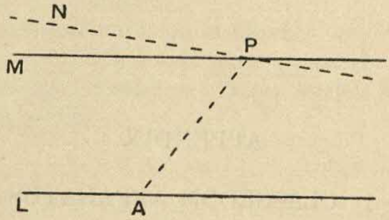


FIG. 96.

On AL take any point A. Join PA, and through P draw a line PM, making the alternate angles at P and A equal. Then by Theorem V the lines AL and PM are parallel.

If possible, draw through P another line PN also parallel to AL. Then by Theorem X the angle NPA is equal to the alternate angle at A, and therefore equal to the angle MPA, which is impossible. Hence no line through P can be drawn parallel to AL except the line PM.

Theorems XII, XIII, XIV and XV deal with the conditions for similarity of triangles: those who wish to pursue the subject further are referred to the original.

It may however be remarked here that the author intends his sequence of theorems to be the logical completion of a scheme of geometry organised around the related ideas of the congruence and similarity of figures: i.e. a scheme in which the study and use of similar figures plays, from the first, a very important part. He uses the properties of a spherical surface, much as they are used on p. 123 of this report, to demonstrate that the existence of similar figures cannot be inferred from the existence of congruent figures, but involves an independent principle.

APPENDIX

1 CLASSROOM APPARATUS

The principal apparatus of mathematical teaching is a blackboard and a box of chalks. But the chalks should be of many colours, because it is worth while to attract the juvenile eye, both for tactical and aesthetic reasons. Strictly speaking, nothing else is necessary, because in geometry, as in mechanics, the appeal should be rather to the pupils' own experience of space-relationships than to artificially contrived experiments. For this reason, the classroom itself and its furniture provide many useful illustrations of the right sort, especially for solid geometry. Two or three rulers, a rectangular box and a mark-book or a sloping desk-lid can also be used effectively. It is a good thing to have a collection of models of the commoner solids, made, preferably, by past and present members of the form, either of wood or paper.

It is worth noting that the makeshift compasses consisting of a piece of string (or the blackboard duster) may convey to a form of beginners a better idea of the circle as a locus than does the more orthodox instrument. When circles of large radius are to be drawn on the blackboard, the piece of string supersedes the compasses in any case.

Beyond this, every teacher will have his own fancy. The following list is doubtless far from complete. Most of the items can and should be made by the boys themselves. None of them is essential.

Slate surfaced globe, with or without lines of latitude and longitude.

Home-made protractors and set-squares, clinometers and diagonal scales.

Meccano strips, etc. (to show the rigidity of the triangle and that the area of a parallelogram does not depend only on the lengths of the sides).

Apparatus to show loci of points on moving cranks and wheels; trammel; pantograph.

Apparatus for teaching plan and elevation : hinged boards for horizontal and vertical planes.

Corks and knitting needles for making solids.

Models of solids :

Cube cut into 6 pyramids (with two such cubes one can make a rhombic dodecahedron).

Cube with tetrahedron to fit inside it.

Triangular prism with 3 triangular pyramids to fit inside it.

Cylinder and cone, sectioned : paper models for developments.

String hyperboloid.

2 SINGLE LETTERS FOR ANGLES

It is customary for teachers when working on the blackboard to mark angles known to be equal in distinctive ways, with dots or crosses, or red rings and yellow; and boys when endeavouring to puzzle out a rider do the same thing on their rough figures, but afterwards have to write out their solution formally, employing the usual notation.

The use of 'the angle *ABC*' as an abbreviation for 'the angle contained by *AB*, *BC*' goes back to the Greeks. It is now commonly written $\angle ABC$. To what extent must we adhere to this notation?

The advantage which is enjoyed by the teacher demonstrating on the board must surely be shared by the boy, even when he comes to write out his solution. And is there any reason for insisting on the longer notation even when it is bookwork that is being reproduced?

It is not suggested here that the ordinary notation should be abandoned, but that it should be supplemented by the use of the single letter (perhaps Greek) for an angle, with some such conventions as these:

(i) If angles are *given* equal, mark them on the figure as α, α_1 and, at the same time, write on the paper $\alpha=\alpha_1$ (given).

(ii) If the angles are *evidently* equal, again mark them as α, α_1 and write on the paper $\alpha=\alpha_1$ (because $AB=AC$).

(iii) If angles have to be *proved* equal, mark them as α, β and write: To prove $\alpha=\beta$.

(Some will prefer α_1, α_2 to α, α_1.) The advantages of this method and notation will be more easily shown by working some examples in full, and certain cautions will be added as need arises.

Example 1. A chord *AB* of a circle is produced to *T* and *TC* is a tangent from *T* to the circle. Prove $\angle TBC=\angle TCA$.

Given: *TC* is a tangent.

To prove : $\alpha=\beta+\gamma$.

Proof : $\beta=\beta_1$ (angle in alt. segm.),

and $\alpha=\beta_1+\gamma$ (extr. angle of $\triangle ABC$),

$\therefore\ \alpha=\beta+\gamma$. Q.E.D.

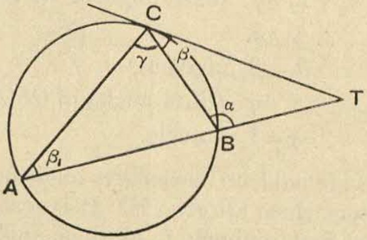


FIG. 97.

Caution. The angle ACT is divided into two parts by CB. It is nearly always better to mark each part separately ; in the first place the process of solution often demands consideration of both parts, and in the second place it is usually difficult to mark the whole angle clearly. But see General Remark (2), below.

Example 2. $ABCD$ is a cyclic quadrilateral. DA, CB produced meet at P; AB, DC produced meet at Q. PO, QO bisect the angles APB, BQC respectively. Prove that POQ is a right angle.

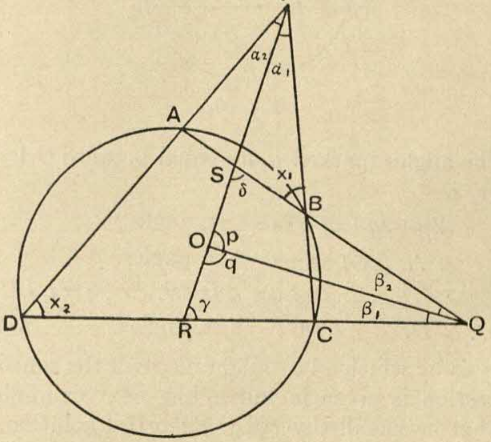


FIG. 98.

Given : $\alpha_1=\alpha_2,\quad \beta_1=\beta_2$.

To prove : $p=1$ rt. angle, which will follow if $p=q$

Proof: $\alpha_1 = \alpha_2$ (given),

$x_1 = x_2$ (cyclic quadl.),

$\therefore \alpha_1 + x_1 = \alpha_2 + x_2$.

But $\alpha_1 + x_1 = \delta$ (extr. angle of $\triangle PBS$),

and $\alpha_2 + x_2 = \gamma$ (extr. angle of $\triangle PDR$),

$\therefore \gamma = \delta$.

But $\beta_1 = \beta_2$ (given),

$\therefore p = q$ (third angles of QSO, QRO),

$\therefore p = 1$ rt. angle.

Caution. Boys are liable to mark every angle in the figure, and to start thinking about them afterwards. It is essential to mark the barest minimum at first, and only to mark an angle when it is about to be used.

In the next example a slight variant of the above method is illustrated.

Example 3. Prove that the opposite angles of a cyclic quadrilateral are supplementary.

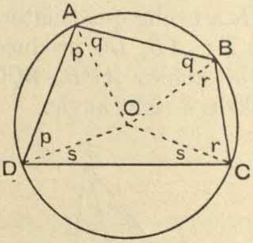


FIG. 99.

Proof: The angles marked p are equal because $OA = OD$. Similarly for q, r, s.

But $2p + 2q + 2r + 2s = 4$ rt. angles,

$\therefore p + q + r + s = 2$ rt. angles.

And $p + q = \angle DAB, \quad r + s = \angle DCB$;

$\therefore \angle DAB + \angle DCB = 2$ rt. angles. Q.E.D.

The analysis by which a boy might discover the solution of a more difficult question is given in full in the next example. It is not suggested that he has discovered the shortest solution.

Example 4. X and Y are the centres of the circles ABP and ABQ; PAQ is a straight line; PX and QY are produced to meet at R. Prove that X, Y, B, R are concyclic.

Analysis. Exterior angles? Angles in the same segment? Angles at X and Y do not look very simple. B is sure to come in. . . . Let us try to prove that $\angle XBY = \angle XRY$. . . . Join AB. Mark α, β, x. Stuck now. . . . α is not subtended by anything . . . but it is at an end of a radius, XB. . . . Join XA. Mark α_1. Similarly with β_1. Only hope for x is to connect it with the other angles of the triangle RPQ. Mark γ, γ_1, δ, δ_1.

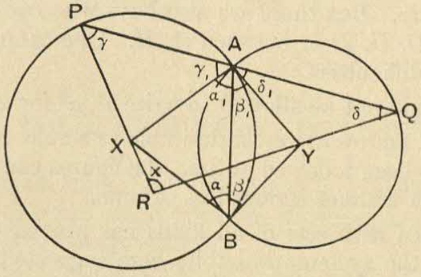


FIG. 100.

(All this has been done mentally of course ; but the time has come to write down what is known so far, in the hope that something else will turn up.)

Solution. To prove : $\alpha+\beta=x$.

Proof:

$$\alpha=\alpha_1 \quad (XB=XA, \text{ radii}),$$
$$\beta=\beta_1 \quad (YB=YA, \text{ radii}),$$
$$x+\gamma+\delta=2 \text{ rt. angles (angles of } RPQ),$$
$$\gamma=\gamma_1 \ (XP=XA, \text{ radii}) ; \quad \delta=\delta_1 \ (YQ=YA, \text{ radii}).$$

(Now comes the discovery. Put down the next thing we know),

$$\alpha_1+\beta_1+\gamma_1+\delta_1=2 \text{ rt. angles},$$
$$\therefore\ x=\alpha+\beta. \qquad \text{Q.E.D.}$$

General Remarks. (1) The real advantage to the boy is gained

(*a*) by writing out the solution bit by bit as he discovers it on the figure ;

(*b*) by having his attention focussed. He knows that he has to prove $\alpha=\beta$, and so he looks at α to see with what other angle he can connect it ;

(*c*) by the fact that the written equations stand out so much more clearly in this notation. If, say, $x_2=\gamma_1+\beta_2$ and $x_1=\alpha_2+\beta_1$ have been proved, it becomes obvious that $\gamma_1=\alpha_2$. If the equations had been $\angle PQR=\angle ABC+\angle XBY$, $\angle PRQ=\angle YAC+\angle BYX$, where $\angle PQR=\angle PRQ$ and $\angle XBY=\angle BYX$, the fact that $\angle ABC$ is therefore equal to $\angle YAC$ does not leap to the eye so readily.

(2) It is often simpler and therefore more desirable to use $\alpha, \beta, \ldots$ for certain angles in a figure, and yet to refer to another one as $\angle PQR$.

(3) There are of course boys who cannot distinguish between y and γ, and so these angles should perhaps not be used in the same figure. But there are also boys who cannot distinguish between O, U, D or between A, H. One cannot legislate for all these difficulties.

(4) Boys should not be allowed to write α^1, α^2 for α_1, α_2.

(5) Freehand figures are most desirable as a rule, for if the wrong track has been followed at first, the figures can be replaced by a new one without serious loss of time.

(6) Experience with sets of all kinds has proved that once boys get into the systematic habits here suggested they have far greater success with riders than they dreamed possible ; and that if they draw large clear figures and put the letters close to the points and in the angles that they represent, the labour of reading their solutions is enormously reduced. The judicious thickening of certain lines in the figure is often a great help to the boy as well as to the reader.

Boys should be taught this method from the very beginning, and indeed should learn to represent an angle by one letter before they represent it by three. (This is stressed in some recent elementary books, though without the extensions here suggested.)

As explained above, there is a certain technique to be learnt ; but the advantages of the method are so great that it is worth while to take some trouble to teach the technique thoroughly.

3 SOLID GEOMETRY; DRAWING

Perhaps the most important object aimed at in Stage A work is the awakening of geometrical intuition or perception, especially in regard to our surroundings in life.

We live in a three-dimensional world and most of our experience is three-dimensional. Plane geometry is already an abstraction and in fact an abstraction which eliminates very much, and that the most interesting part, of our experience. Hence it seems obvious that the more completely the introductory work can be three-dimensional the richer, the more interesting and the more valuable it should prove.

There is a further consideration, both important and practical, viz. that the longer three-dimensional work is deferred the harder does it become to get over its fundamental difficulty. This seems to be because exclusive familiarity with the figures of plane geometry hinders rather than assists three-dimensional intuition, and especially the power of reading and making drawings of three-dimensional facts.

No one probably disputes the value of actual construction of some of the simpler solids from their nets, and perhaps the illustrations given in pp. 35-40 of the present report will convince many that great benefits are derived from drawing the solids; the questions remain how far these two processes, construction and drawing, can be carried with advantage, having regard to the limited time available, and whether they can be extended so far as to cover all that it is desirable to include in Stage A.

The latter question is the more important and in effect the answer to it will more or less include the answer to the former; so we will consider it first.

It will probably be admitted that the most substantial and the most interesting part of Stage A work is found in 'Heights and Distances' and that the more interesting of these are those which concern three-dimensions; and further that the difficulty of these lies in their graphical representation—so we are back at the fundamental difficulties of drawing and intuition.

There is a difference, small in itself, but needing careful treatment, between drawing a cube or any single solid figure and drawing several objects in their relative situations in a free space. The principles are all involved in the drawing of a cube, but they now have to be restated and practised from a wider point of view.

The principles amount to those requisite for drawing and using three-coordinate axes ; the practice involves search for questions suitable for their illustration.

As to the former point we should not of course begin by talking about coordinate axes but avoid technical terms and base the work on familiar experience and observation. A suitable introduction is as follows :

'Draw the corner of the room.' This demand will probably produce such drawings as

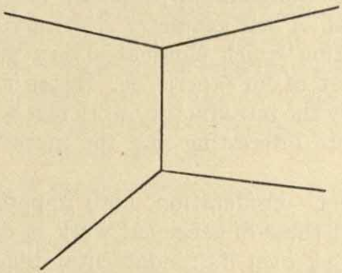


FIG. 101.

' Now simplify this by omitting the ceiling and by drawing the end wall as if you were looking directly at it and not at the corner '. The result should be :

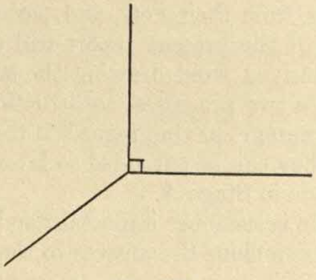


FIG. 102.

' Put a door in the end wall.' It will be its natural shape. ' Put in the floor boards.' If they are put in converging have the convergence removed.

' Now suppose the floor divided up like a chess board ; how will you put in the cross lines? '

The final figure will be

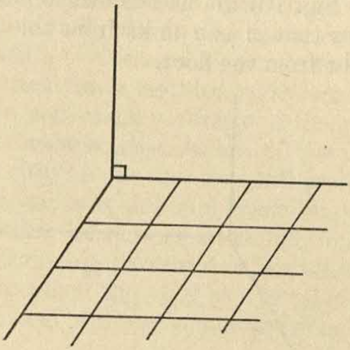


FIG. 103.

If we put a chess board pattern on the end wall the squares would be represented by actual squares, i.e. the vertical and the right and left scales will be the same. The squares on the floor and on the side wall will be parallelograms; the scale along the back to front lines arbitrary but uniform and for convenience less than the vertical and cross scale.

The angle between the two sides of the floor is also arbitrary. For general use this may conveniently be 150° and the one scale say half the other; or perhaps for greater convenience 1″ for the one, 1 cm. for the other.

'If now having drawn my corner (or my axes if the term is preferred) I merely mark a point or draw a line on the board, does that mark tell me where the point or the line really is?' Clearly not.

Some practice is therefore necessary in representing known points and lines and conversely in interpreting the more or less conventional representation in a drawing.

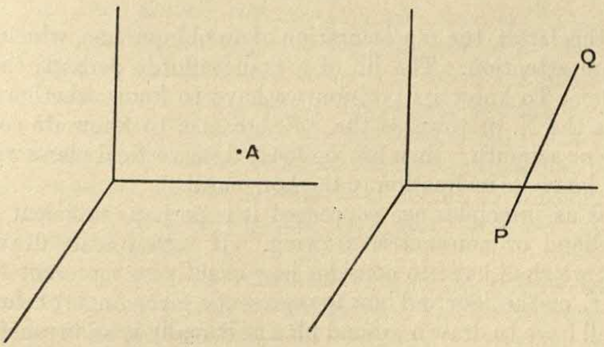


FIG. 104.

The basic facts are :

The diagrams of Fig. 104 are useless and almost meaningless.

Fig. 105 indicates that A is a units from the end of the room, b from the side and c from the floor.

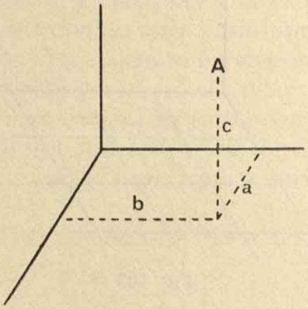


FIG. 105.

Fig. 106 indicates that PQ is a line whose compass bearing is that of PN, and whose inclination to the horizontal is shown, but in perspective, by QPN. Incidentally QNP represents a right angle.

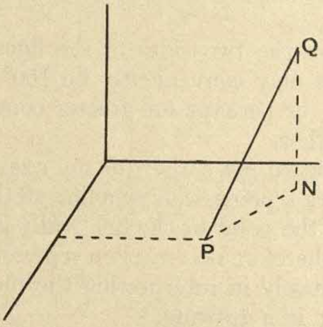


FIG. 106.

It is this latter, the representation of an oblique line, which needs the more attention. The jib of a crane affords perhaps the best examples. To know its position we have to know whether it lies towards the N. or towards the S.E. etc., i.e. to know its compass bearing or azimuth ; then having located its vertical plane we must further know its inclination to the horizontal.

So far as principles are concerned it is perhaps sufficient to rely on freehand or non-metric drawing. If actual scale drawing is wanted we shall have to consider how exactly to represent N.E. or N. 60° E. on the floor and how to represent a given angle of elevation. We shall have to draw a ground plan as it really is and transfer it to the oblique representation of the perspective drawing. This is in itself most valuable work involving, but in a richer environment, the exercises in triangulation which form a basic element in Stage A. It leads on quite naturally also to plan and elevation drawing.

For the cultivation of this kind of perspective representation an excellent field of examples is afforded by the study of shadows.

AB represents a post stuck vertically in the ground, *L* is a light on the end wall: construct the shadow of the post. Anyone who proposes this to a class for the first time will probably be astonished at the abysmal ignorance or lack of geometrical perception in regard to presumably familiar facts displayed by his pupils. Even that the line *LB* is required may be perceived only with some difficulty, but the crux is in perceiving that *MA* also is required. Considerable discussion, apart from the immediate problem, will probably be required. Such questions as the following will be found useful: How must you stand with regard to a light so that you may be facing along your shadow? If the sun is in the S.E. how does your shadow lie? Are the sun shadows of two vertical posts parallel? Are those thrown by a street lamp parallel?

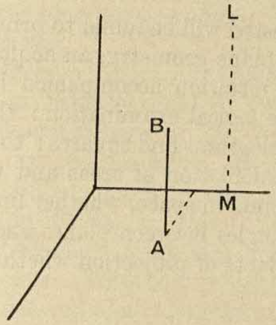


FIG. 107.

When the basic problem, to determine where *LB* meets the ground, has been solved, suitable examples are easily multiplied: shadows of a pair of goal posts, of the gable end of a house, of the jib of a crane. Two cases need to be dealt with, sun shadows and shadows thrown by a lamp; and in one respect at least the former may give the more difficulty, for the sun itself is not in the picture, and there is nothing corresponding to the foot of the lamp.

Besides shadows on the ground it is useful to consider those thrown partly on the ground, partly on a wall; or those thrown on a sloping embankment.

Along with this work the construction of solids and especially the drawing of them may well be continued. The following may be mentioned as certainly worthy of attention: pyramids with a triangular base and the vertex over a specified point in the base, a non-rectangular parallelepiped, a shed built against a wall with a sloping roof, and in particular an extension of the last which will be found most instructive, viz.:

Two garden walls meet at right angles; in the corner a greenhouse is built with two wings one against each wall; the dimensions are given, i.e. length and breadth of each wing and heights at back and front. Make a sketch in parallel projection and a model.

In such questions it will be found essential to draw the ground plan in parallel projection first, and then to put in the necessary vertical lines and complete the figure.

Arithmetical questions can of course easily be based on the structure, and it is quite worth while and justifiable to teach the rule for calculating the third side of a right-angled triangle when two sides are given.

The work here suggested will be found to bring acquaintance with the following parts of plane geometry, an acquaintance arising in a natural way through intuition accompanied by real thought, although without precise logical examination: the fundamental facts about triangles, parallelograms and squares; the intercept theorem; similar figures; the calculation of areas and volumes; properties of lines, planes and points in space, whether lines intersect, where a line meets a plane; angles between planes and between a line and a plane; the simpler facts of projection whether by parallel rays or by rays from a point.

4 THE THEORY OF DRAWING

The very simple method for the representation of solids advocated in this report is much used in practice but often without any clear notion of its scientific basis. Its German name is Axonometric; English seems to have no equivalent word, and we have to call it oblique parallel projection.

Its fundamental theorem Klein says (*Elementar-Mathematik*, II, 89) was discovered in 1853 by Pohlke, Professor of Descriptive Drawing in the Building Academy of Berlin, and published in 1860 in his *Text Book of Descriptive Geometry*. It is to the effect that unit lengths along three rectangular axes can be represented by *any* three radiating lines on the paper. Parallel lines of course remain parallel and in each direction there is a uniform scale, e.g. mid-points remain mid-points. Thus to represent a cube we can draw three lines at any angles and of any lengths and complete the figure by drawing parallels.

A figure so drawn is in fact the shadow of the solid thrown on an arbitrary plane by sunlight; how far it looks like the solid depends on the degree of obliquity of the paper to the light; and it will only look right if the paper is approximately at right angles to the light, i.e. if the projection is approximately orthogonal. But to draw a true orthogonal projection requires a degree of care which is out of place when we are drawing figures rapidly for geometrical purposes; two of the axes and their units can be drawn at random, but the third is then determinate and not too easily determined. Hence it is better, for geometrical purposes, to be content with the more general oblique projection with its three arbitrary axes. When one speaks of a 'freehand sketch' one probably means such a projection with the axes so chosen as to look approximately right, i.e. a projection which is approximately orthogonal. But a 'freehand sketch' may also imply an attempt at perspective proper in which parallel lines converge; the drawback then is that rules of measurement become complicated, e.g. mid-points do not remain mid-points, a grave disadvantage when the original figure has to be developed into something more complicated.

The proof of Pohlke's theorem falls into two parts, the determination of the direction of the light and of the orientation of the plane of projection required to produce the figure chosen.

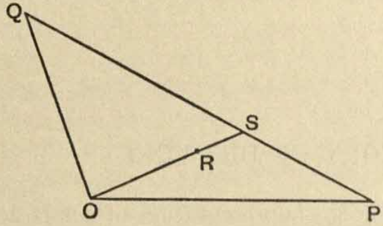


FIG. 108.

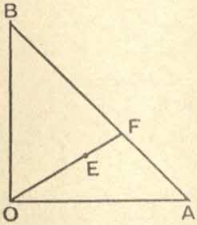


FIG. 109.

[N.B. $AF : FB = PS : SQ$; and $OE : EF = OR : RS$.]

Let OP, OQ, OR represent three concurrent edges of a cube, OA, OB, OC. The point R represents all points in the ray through C and in particular the point where this ray meets the face OAB. Calling this point E, the direction of the light with respect to the cube is determined by CE. We may then think of the cube as strung on this line CE and the rays passing through the sides of the triangle OAB as forming a triangular prism of given cross section.

The problem then becomes that of determining the orientation to the axis of this prism of a plane which produces the section OPQ. There does not seem to be any simple geometrical solution of this, but with the help of algebra or trigonometry it is easily found that there always is a real solution and that it is unique except that there are two positions symmetrically related to the axis.

We have just seen that any triangle can be projected into any other triangular shape when the direction of the rays relative to the original triangle is given. There is another similar proposition which is useful. It may be stated in the form: the patch of light thrown on the floor by a triangular window can take any given triangular shape and the direction of the light required to produce a given shape from a given window is determinate and unique.

The proof of this is simple.

The triangle ABC on the wall is to be projected into a triangle PQR on the floor so that $\angle P = \alpha$, $\angle R = \gamma$. Let the sides of triangle ABC meet the line of intersection of the floor and the wall in H, K, L. Through C draw CM parallel to AB meeting this line of intersection in M.

On HK and KM describe segments of circles containing the angles γ and α. These meet in R the projection of C.

Through A and B draw parallels to CR meeting KR and HR in P and Q respectively: PQR is the triangle required.

For since AP, BQ, CR are parallel, the intersection of corresponding sides are collinear (Desargues' Theorem), $\therefore$ PQ goes through L.

But
$$QR : RH = BC : CH = LM : MH,$$
$$\therefore\ QL \text{ is parallel to } RM,$$
$$\therefore\ \angle P = MRK = \alpha.$$

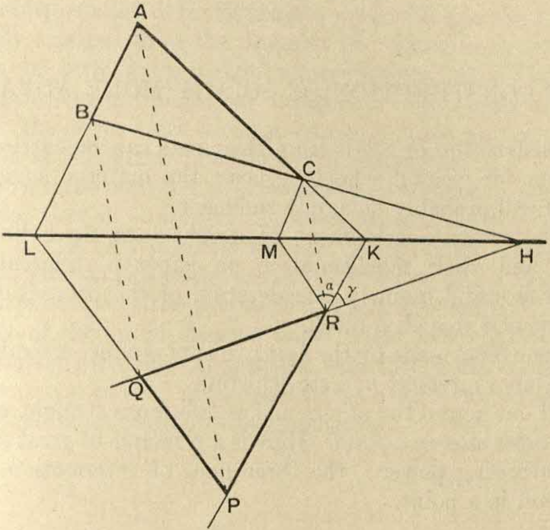


Fig. 110.

Whatever the angle between the two planes, here spoken of as wall and floor, the positions of P, Q, R in their plane will be unaltered and the proper form of the proposition is: any triangle can be uniquely projected into a triangle of a given shape when the direction of the axis of projection (i.e. the intersection of the two planes) is given.

5 THE CONSTRUCTION OF SOLIDS (MORE ADVANCED)

The construction of solids from their nets can be carried to any length, but for general school purposes the material suggested on pp. 37-40 will probably be amply sufficient.

For the further development of insight into three-dimensional relations and their representation on paper a different line of approach is useful, namely consideration of the process of sawing a solid out of a shapeless block.

The plane faces made by the first two cuts produce a straight edge, i.e. two planes intersect in a straight line.

A third cut across the others makes two more straight edges and the *three edges meet in a point.* Here is a principle of great simplicity but of surprising power: the three lines of intersection of three planes meet in a point.

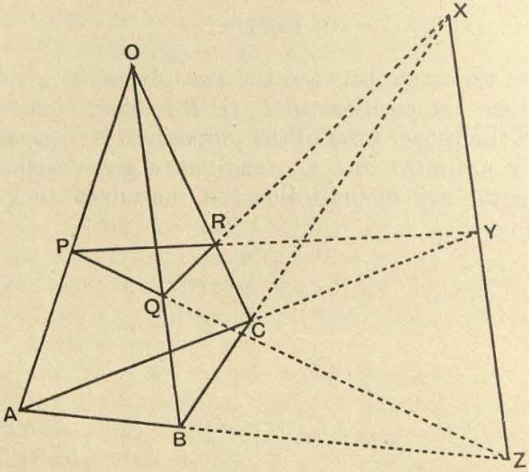


FIG. 111.

A fourth cut across the other three produces a pyramid *OABC* (Fig. 111). But now suppose the material actually cut off were still in place so that it was possible to make a fifth cut *PQR* across the pyramid and carry it on till it met the plane ABC; we then get a figure of extreme importance, that of Desargues' theorem.

The last cut PQR meets the base ABC in a straight line XYZ and the three faces of the pyramid meet this line each in a point; hence BC and QR, CA and RP, PQ and AB meet in three collinear points X, Y and Z.

The interpretation of the figure as a proposition in plane geometry commonly referred to as the property of 'triangles in perspective' can only be proved by metric considerations, while the theorem itself depends on properties of 'incidence' only, especially on the fact that the three lines of intersection of three planes meet in a point, and is altogether independent of measurement.

The very term used in the proposition, 'triangles in perspective' is evidence that it is properly a three-dimensional proposition, for 'perspective' is only properly relevant to three-dimensional space.

The proposition lies at the root of all drawing in perspective.

The importance of the proposition is so great compared with the simplicity of the proof that its value might easily be overlooked. An alternative method of approach is therefore worth consideration. Let this exercise be set; make from its net a solid with three irregular quadrilateral faces and two triangular ends.

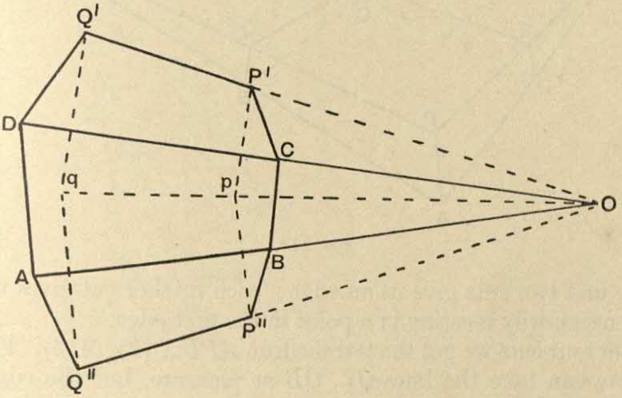


Fig. 112.

One face $ABCD$ can be laid down at random (Fig. 112): can the next $DCP'Q'$ also? No; for let $Q'P'$ meet DC in O' and $Q''P''$ meet AB in O'', then as $Q'P'$ and $Q''P''$ are folded up round DC and AB respectively, they will continue to meet the plane $ABCD$ in O' and O'', and unless these two points coincide $Q'P'$, $Q''P''$ cannot be brought into coincidence. Therefore $Q'P'$, $Q''P''$ must pass through O the intersection of DC, AB. When this vital point has been recognised the completion of the figure is a straightforward exercise involving only principles which should have been previously acquired.

A pyramid with a polygonal base can of course be treated as is the triangular pyramid of Desargues' theorem; and this affords excellent practice in drawing and helps to impress the general principle that two plane figures in perspective have an axis of perspective as well as a vertex. An important special case is that in which the plane of section is parallel to the base and we get the ' spider web ' figure of similarity (p. 91).

If two *plane* polygons are in perspective, i.e. if one exactly covers the other when seen from a point V, they also have an axis of perspective. Is the converse true? i.e. if the intersections of pairs of corresponding sides are collinear are the joins of corresponding vertices concurrent? It is easily seen that except in the case of triangles this converse is not necessarily true.

Further valuable exercise in drawing and insight into facts is afforded by consideration of the process of cutting a solid bounded by six irregular quadrilaterals from a shapeless block.

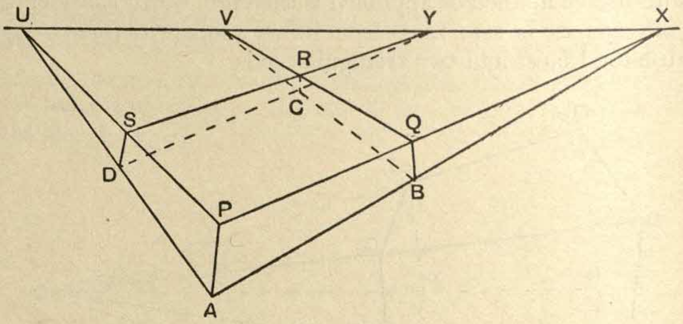


FIG. 113.

The first two cuts give us an edge; each further cut gives us two edges necessarily meeting in a point in the first edge.

After four cuts we get the tetrahedron $APUX$ (Fig. 113). For the fifth we can take the lines QV, QB at pleasure, but the edge BV follows of necessity. Similarly for the last, we can take SR, SD at pleasure, but the cut (if the material were still there) would meet UVX in a point Y and the edge DC follows of necessity and therewith the eighth corner C.

This figure in its general form is of no particular geometrical importance, but the construction of such a solid from its net is a most interesting and instructive piece of drawing. The use of the ' axis ' $UVXY$ is of course essential.

But there are two special cases which are of fundamental geometrical importance.

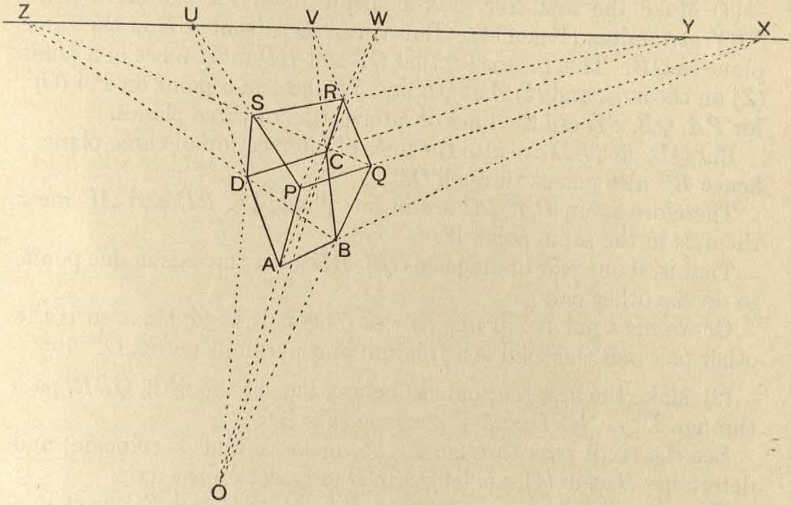


FIG. 114.

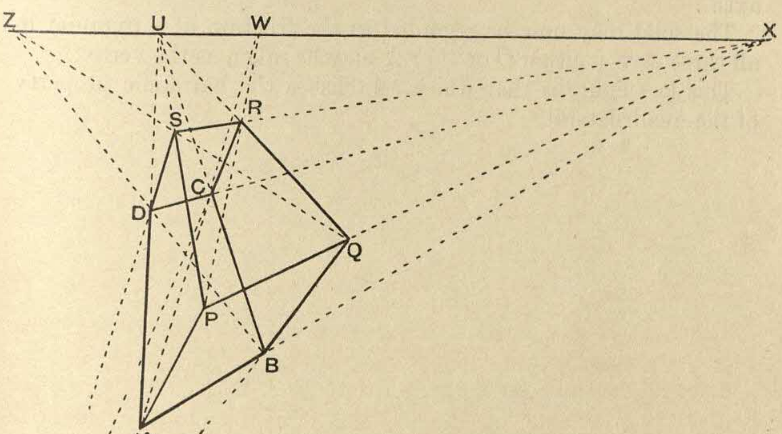


FIG. 115.

(1) Make the first five cuts as before, and for the sixth draw *SRY* at random (Fig. 114). But draw *SD* so that it is in the same plane as *QB*. This means (*a*) that *QS* and *BD* must meet in a point (*Z*) on the axis, and (*b*) that *QB* and *SD* meet in a point on *PA* (*O*); for *PA*, *QB*, *SD* are the lines of intersection of three planes.

But *QB*, *RC*, *SD* are also the lines of intersection of three planes; hence *RC* also passes through *O*.

Therefore again *PA*, *RC* are in one plane, i.e. *PR* and *AC* meet the axis in the same point *W*.

That is, if one pair of diagonals *QS*, *BD* meet the axis in one point, so do the other pair.

Or we may put it: if one pair of diagonals is coplanar so is the other pair and the solid is a frustum of a pyramid vertex *O*.

(2) Make the first four cuts as before, but let the fifth, *QBR*, pass through *U*, i.e. let *U* and *V* coincide (Fig. 115).

Let the sixth pass through *X* (i.e. make *X* and *Y* coincide) and determine *D* as in (1), i.e. let *QS*, *BD* intersect on the axis.

Then it will follow as in (1) that *PA*, *QB*, *SD* and *RC* meet in a point *O* and that the other pair of diagonals *PR*, *AC* meet on the axis.

The solid may now be regarded as the frustum of a pyramid in all three ways; either *O* or *U* or *X* may be taken as the vertex.

This last figure is that which establishes the harmonic property of the quadrilateral.

6 THE THEOREM OF PYTHAGORAS

According to tradition it was Pythagoras (*c.* 550 B.C.) who discovered the truth, in the general case, of the theorem about the squares on the sides of a right-angled triangle ; this however is uncertain.

Special cases, such as that illustrated in Fig. 116 and that of the 3, 4, 5 triangle, were known much earlier by the Egyptians and Chaldeans and in China (*c.* 1100 B.C.).

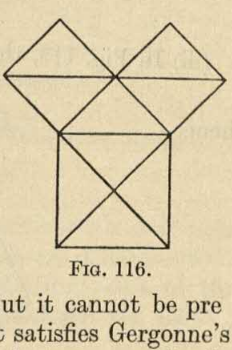


FIG. 116.

It is implied by Proclus that the proof in *Euclid's Elements*, with which English teachers are still familiar, was due to Euclid himself, and it is to be hoped that teachers will not allow that classical proof to be forgotten. But it cannot be pretended that Euclid's proof is the one that best satisfies Gergonne's criterion of being easy to explain in a few words to a passer-by in the street. Many alternatives to that proof have been given, and without attempting to decide which proof comes nearest to that ideal, and still less to decide which is the best proof for examination purposes, we will give a few indications of miscellaneous methods.

A number of proofs, including Euclid's, depend on showing that $BA^2=BD\,.\,BC$, using the similar result, and adding. Amongst these, three may be mentioned :

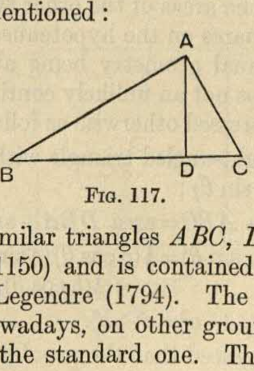


FIG. 117.

(i) The proof by similar triangles ABC, DBA. This was given by Bhaskara (*c.* A.D. 1150) and is contained in many Continental geometries following Legendre (1794). The early introduction of similarity is urged, nowadays, on other grounds ; it is a necessity if this proof is to be the standard one. The same applies to (ii) and (iv).

(ii) The similar triangles BAZ, BYA in Fig. 118 give

$$BA^2 = BY \,.\, BZ.$$

And
$$\begin{aligned} BY \,.\, BZ &= (BC+r)(BC-r) \\ &= BC^2 - r^2, \end{aligned}$$

$$\therefore\ BA^2 = BC^2 - CA^2.$$

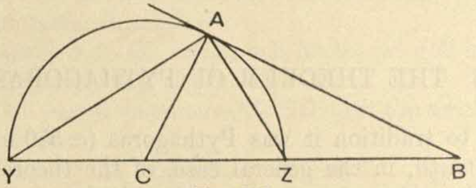


FIG. 118.

(iii) In Fig. 119, the triangles KNB, ADB are congruent,

$$\therefore\ BN = BD,$$

hence
$$\begin{aligned} BD \,.\, BC = BN \,.\, BC &= \text{area } KMCB \\ &= \text{area } KHAB = BA^2. \end{aligned}$$

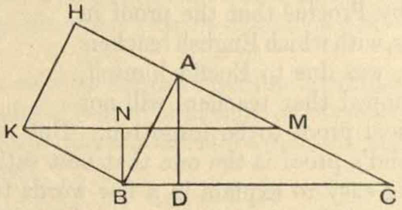


FIG. 119.

Another similarity proof makes use of the property that the areas of similar triangles are proportional to the squares on their corresponding sides, as follows:

(iv) ABC, DBA, DAC in Fig. 117 are similar, and the area of the first is the sum of the areas of the other two; also the areas are proportional to the squares on the hypotenuses.

In the event of formal geometry being abandoned in the near future, which is perhaps not an unlikely contingency, this last proof would probably be expressed otherwise as follows:

(v) The area of a right-angled triangle with hypotenuse a and an angle θ is $\frac{1}{2}(a\cos\theta)(a\sin\theta)$;

but
$$\text{area } ABC = \text{area } DBA + \text{area } DAC,$$

$$\begin{aligned} \therefore\ \tfrac{1}{2}(a\cos\theta)(a\sin\theta) &= \tfrac{1}{2}(c\cos\theta)(c\sin\theta) \\ &\quad + \tfrac{1}{2}(b\cos\theta)(b\sin\theta), \end{aligned}$$

$$\therefore\ a^2 = c^2 + b^2.$$

(vi) It has been suggested that the proof given by

$$(a+b)^2 = c^2 + 2ab$$

in Fig. 120 may have been the proof known to Pythagoras, but Sir T. Heath has suggested that a similarity proof (so far as then available) is more likely.

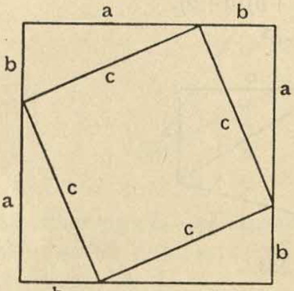


FIG. 120.

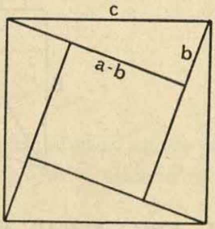


FIG. 121.

(vii) A proof of the same kind was given by Bhaskara, using

$$c^2=(a-b)^2+2ab, \text{ in Fig. 121.}$$

(viii) In Fig. 122 there are four congruent quadrilaterals and hence two hexagons, *KHGFCB* and *ABELDC*, which are equal in area; the proof is obtained by subtracting $2\triangle ABC$ from each of the hexagons. This proof has been attributed to Leonardo da Vinci.

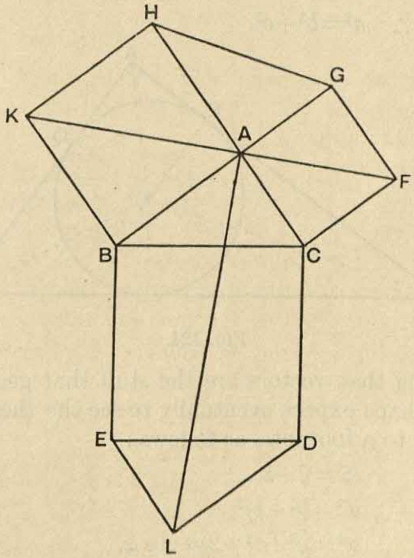


FIG. 122.

(ix) (President Garfield's proof.) Fig. 123 is really one half of Fig. 120.

By expressing the area of the trapezium in two ways:

$$ab+\tfrac{1}{2}c^2=\tfrac{1}{2}(a+b)(a+b),$$
$$\therefore\ c^2=a^2+b^2.$$

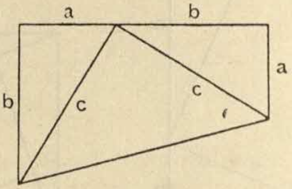


FIG. 123.

(x) If ABC, $A'B'C'$ are two triangles whose correspondingly lettered sides are equal and parallel, it is easy to show that two of the parallelograms formed by joining correspondingly lettered corners are together equal in area to the third. The theorem follows by taking $BB'=BC$ and $\angle BAC=1$ rt. angle $=\angle B'BC$.

(xi) In Fig. 124,

$$\tfrac{1}{2}bc=\triangle=\triangle BIC+\triangle CIA+\triangle AIB$$
$$=\tfrac{1}{2}r(a+b+c),$$

and
$$r=AP=AQ=\tfrac{1}{2}[(a+b+c)-2a],$$
$$\therefore\ 2bc=(a+b+c)(b+c-a),$$
$$\therefore\ a^2=b^2+c^2.$$

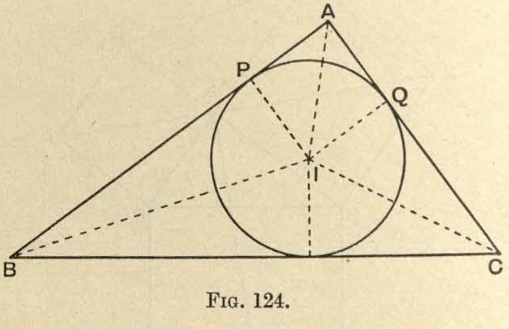


FIG. 124.

(xii) Assuming that vectors are the stuff that geometry is made of, we may perhaps expect eventually to see the theorem of Pythagoras relegated to a footnote, as follows:

In Fig. 125, $\bar{x}=\bar{y}+\bar{z}$,

$$\therefore\ \bar{x}^2=(\bar{y}+\bar{z})^2,$$

i.e. $x^2=y^2+z^2+2yz\cos\alpha$.

Note: if $\alpha=\tfrac{1}{2}\pi$, $x^2=y^2+z^2$.

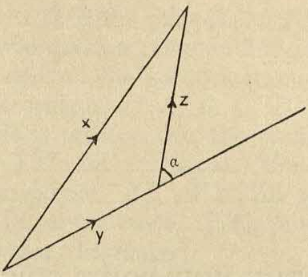


FIG. 125.

For some other proofs and historical details reference may be made to Heath's *Euclid*, Vol. I and to E. Fourrey's *Curiosités Géométriques* (Vuibert, Paris), Ch. II.

7 THE NINE-POINT CIRCLE

The property of the nine-point circle of a triangle was discovered by Brianchon and Poncelet in 1820. In the original proof it was shown by the similar triangles ABE, ACF in Fig. 126 that

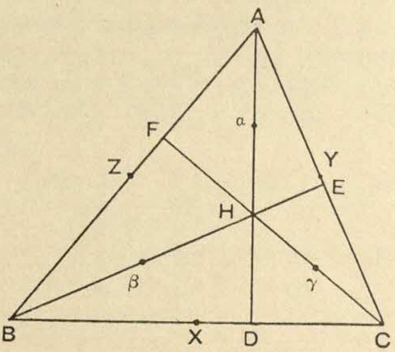


FIG. 126.

$$AB \,.\, AF = AC \,.\, AE. \quad \therefore\ AZ \,.\, AF = AY \,.\, AE.$$

$$\therefore\ Y, Z, E, F \text{ are concyclic.}$$

Similarly for Z, X, F, D and for X, Y, D, E.

The three circles coincide, because otherwise their common chords must be concurrent, whereas these are actually the sides of the triangle.

Thus the six points X, Y, Z, D, E, F lie on a circle.

Just as in the triangle ABC it has been shown that X, Y, Z lie on the circle DEF, so it might be proved in the triangle HBC that β, γ, X lie on the circle DEF, and so the theorem is completely proved. Actually in the original this second part was given in full.

The excuse for the more complicated proofs that are sometimes given in text-books is that they avoid the assumption about the concurrence of common chords of three circles.

One of the most satisfactory elementary proofs is the one that regards the triangle and its orthocentre as a quadrangle (4-point).

For any quadrangle $ABCD$ the joins XX', YY', ZZ' of the mid-points of opposite sides have a common mid-point O, which is, in fact, the centre of each of the parallelograms $YZY'Z'$, $ZXZ'X'$, $XYX'Y'$ and is the centroid of A, B, C, D for equal multiples. When the quadrangle is orthocentric, the parallelograms are rectangles and so XX', YY', ZZ' are equal. Hence a circle centre O passes through the six points. XX', YY', ZZ' are diameters and so P, Q, R in Fig. 127 lie on the circle. [This proof can, of course, be stated in more elementary language.]

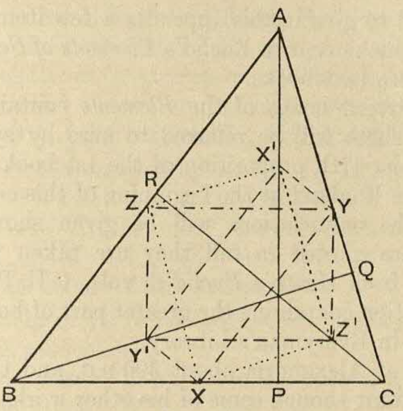


FIG. 127.

From a higher standpoint the mathematician recognises that the nine points are nine of the centres of conics circumscribing the quadrangle ; therefore they lie on a conic. The circumscribing parabolas of an orthocentric quadrangle are also rectangular hyperbolas and their centres are the circular points at infinity ; hence the locus is a circle.

This, like the previous proof, emphasises the fact that the nine-point theorem is a property of a quadrangle rather than of a triangle.

8 EUCLID

It is proposed to give in this appendix a few items of historic or other interest which occur in Euclid's *Elements of Geometry*, but not usually in modern text-books.

There are thirteen books of the *Elements* containing in all 465 propositions. These will be referred to here by symbols such as I 47, meaning the 47th proposition of the 1st book. This was the usual practice in England at the beginning of this century.

Sometimes the enunciations will be given shortly in modern notation; where quoted in full they are taken with occasional verbal changes from Heath's *Euclid* (3 vols., C.U. Press).

A smaller edition containing the greater part of books I to VI and XI is available in Everyman's Library.

Euclid lived at Alexandria about 300 B.C. and the whole of his *Elements* are extant though some of his other works have been lost. The so-called books XIV, XV which have been included in some editions are probably not due to Euclid. Euclid of Alexandria is to be distinguished from the philosopher Euclid of Megara, who lived about 100 years earlier.

The propositions of book I deal with congruence, parallels, and areas, culminating in the theorem of Pythagoras and its converse. They include constructions as well as theorems. Euclid's constructions serve as existence theorems. I 46 shows that a figure actually exists having the properties attributed to a square in the definition of that figure, and in III the existence of a unique tangent to a circle at a given point is established. In I 9-12, constructions are given for bisectors and perpendiculars. Euclid did not use hypothetical constructions, e.g. he did not use bisectors or perpendiculars in I 1-8.

Book I begins with 23 definitions, 5 postulates, and 5 common notions. The common notions include:

> Things which are equal to the same thing are also equal to one another.
>
> The whole is greater than the part.

One of the construction postulates and the parallel postulate may be quoted:

Let the following be postulated. . . .

3 To describe a circle with any centre and radius.

5 That if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

This postulate 5 was called axiom 12 in some modern editions. It is now usually replaced by Playfair's axiom. Postulate 3 is commonly interpreted to mean that a circle can be drawn with centre A and radius AB where AB is any straight line drawn from A. It is not assumed at first that a circle can be drawn with centre A and radius BC. Thus the use of compasses is demanded but they are not to be used to transfer distances. In I 2 however it is shown how to draw a circle centre A and radius BC ; the construction is :

> draw an equilateral triangle ABX (by I 1) ; produce XB to meet the circumference of the circle centre B, radius BC, at Y ; produce XA to meet the circumference of the circle centre X, radius XY, at Z ; then the circle required can be drawn with centre A and radius AZ.

Thus with the help of I 2 the disadvantage of the restricted use of the compasses is overcome.

Two of the propositions of book I which occur only as exercises in most modern geometries are

I 21 $BP+PC<BA+AC$ and $\angle BPC>\angle BAC$. See Fig. 128.

BP is produced to meet AC at Q. It was proved in I 20 that any two sides of a triangle are together greater than the third, and so it is easy to show that $BA+AC>BQ+QC$ and similarly that

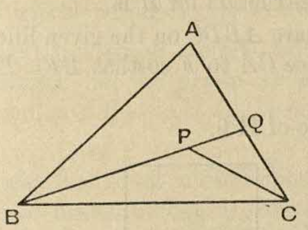


FIG. 128.

$$BQ+QC>BP+PC.$$

I 43 In any parallelogram, the complements of the parallelograms about the diagonal are equal to one another.

In connection with this, see Fig. 35. I 43 is a stage of the process by which Euclid shows how to construct a square equal to a given rectilineal figure, a process completed in II 14. It is easy to make a parallelogram equal to a given triangle (I 42) and parallelograms can be made equal to the two parts into which a quadrilateral is divided by its diagonal. Euclid's solution of the problem of fitting the two parallelograms together is contained in I 43, 44; it is generally solved otherwise to-day, and so I 43 tends to drop out of sight. This is unfortunate, as it has a value of its own apart from the use that Euclid makes of it.

Book II begins with the dissection of a rectangle into rectangles. This dissection and indeed the existence of rectangles themselves depends upon Euclid's parallel postulate. It is a consequence of the postulate that a plane can be divided into equal regular figures of 3, 4, or 6 sides and of any size, whereas the surface of a sphere cannot be divided indefinitely into congruent portions of any kind whatever although it can be divided into 4, 8, or 20 equal triangles.

Euclid gives in book II careful proofs of geometrical theorems which correspond to algebraic identities such as

$$x(a+b+c)=xa+xb+xc, \quad (a+b)^2=a^2+2ab+b^2.$$

II 6 corresponds to $x^2-y^2=(x+y)(x-y)$,
where $x=CA$, $y=EA=AF$ in Fig. 129.

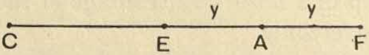


FIG. 129.

In II 11 there is a construction for the point H in AB such that $AH^2=BH\,.\,BA$, an essential part of the construction of the regular pentagon. The construction for H is

> draw the square $ABDC$ on the given line AB; bisect AC at E and produce CA to F so that $EF=EB$, draw the square FH on AF.

The proof makes use of II 6.

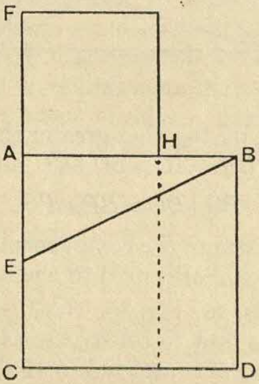


FIG. 130.

This book ends with the two geometrical theorems equivalent to the formula $a^2 = b^2 + c^2 - 2bc \cos A$ and the construction (II 14) of a square equal to a given rectangle. Hence in virtue of what has already been done in Book I, a square can now be constructed equal to any given rectilineal figure.

Books III and IV are concerned with the geometry of the circle. III 1 to 13 include a number of propositions, now omitted from geometry courses, which were essential to Euclid's logical sequence. For example:

if two circles cut, they will not have the same centre,
and a circle does not touch a circle at more points than one.

It may be of interest to quote the theorems dealing with a tangent. These are

III (part of) 16: The straight line drawn at right angles to the diameter from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed. Porism: from this it is manifest that the straight line drawn at right angles to the diameter from its extremity touches the circle.

III 17: From a given point to draw a straight line touching a given circle. (Euclid's construction is independent of the parallel postulate, as also is the whole of III 19.)

III 18: If a straight line touch a circle and a straight line be joined from the centre to the point of contact, the straight line so joined will be perpendicular to the tangent.

III 19: If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to this tangent, the centre of the circle will be on the straight line so drawn.

III 32: If a straight line touch a circle and from the point of contact there be drawn across, in the circle, a straight line cutting the circle, the angles which it makes with the tangent will be equal to the angles in the alternate segments of the circle.

In book IV, after some more elementary constructions, a regular pentagon is obtained (IV 11) from the triangle with angles 72°, 72°, 36° constructed in IV 10 by means of II 11.

Book V is the most remarkable of the whole treatise. It contains the theory of proportion traditionally due to Eudoxus. The discovery of incommensurable magnitudes was made by the early (pythagorean) Greeks; prior to that there was a theory of proportion applicable to rational numbers only, and that earlier theory is followed by Euclid in book VII. The definition of proportion, given in V defn. 5, is :

> Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of the latter equimultiples respectively taken in the same order. (Algebraically, $a : b = c : d$ if $mc >=$ or $< nd$ according as $ma >=$ or $< nb$.)

This is used in VI 1 to prove that triangles and parallelograms of the same height are to one another as their bases, and that in turn is required in VI 2 to prove that

> if a straight line be drawn parallel to one of the sides of a triangle, it will cut the other sides proportionally.

This is the foundation of the whole theory of similar figures as given by Euclid in his book VI. V defn. 5 is also used in VI 33 to prove that in equal circles angles have the same ratios as the arcs on which they stand.

The use of inequalities in V defn. 5 corresponds to their use in the modern theory of irrational numbers due to Dedekind.

A short critical account of book V was given by M. J. M. Hill in the *Mathematical Gazette*, vol. xi, p. 213. See also vol. xiv, p. 36.

The foundations of the theory of similarity having been laid down in V and VI 1, 2, the development is then not very different from that in modern geometries. As soon as the properties of similar figures are available, there is, in VI 25, a natural extension of the work in I 42-45 and II 14.

VI 25 constructs one and the same figure similar to a given rectilineal figure and equal to another given rectilineal figure.

The constructions of VI 28, 29 correspond to the solution of the quadratic equations $x(2c \pm x) = b$, where $c^2 \geqslant b$.

The solution of the quadratic equation is implicit also in II 5, with a restriction (b = a square) more important from the ancient point of view than from the modern.

Books VII, VIII, IX may be called the arithmetical books. In VII 3 and 34 are given the usual methods of finding the greatest common measure and least common multiple of two numbers, but most of this book is concerned with the numerical theory of proportion. The propositions of this section correspond closely with those of book V. They include the results which in modern notation would be written

If $a : b = c : d\,(a > c,\ b > d)$, then $(a - c) : (b - d) = a : b$.

If $a : a' = b : b' = c : c' = \ldots$, then

$$(a+b+c+\ldots) : (a'+b'+c'+\ldots) = a : a'.$$

If $a : b = d : e$ and $b : c = e : f$, then $a : c = d : f$.

If $a : b = c : d$, then $ad = bc$.

The treatment in these arithmetical books is a relic of the days before the discovery of incommensurables and the invention of the theory in book V, and it is interesting that Euclid should have reproduced the older theory at this point.

Book VIII deals with geometric progressions, called by the Greeks 'numbers in continued proportion'. The summation, to n terms, of a geometric progression is given in the following form:

> IX 35: If as many numbers as we please be in continued proportion, and there be subtracted from the second and the last numbers equal to the first, then as the excess of the second is to the first, so will the excess of the last be to all before it.

Algebraically, if $a_1, a_2, a_3, \ldots a_n$ are in G.P., then

$$a_2 - a_1 : a_1 = (a_{n+1} - a_1) : (a_1 + a_2 + \ldots + a_n).$$

The proof is equivalent to the following:

$$\frac{a_{n+1}}{a_n} = \frac{a_n}{a_{n-1}} = \ldots = \frac{a_2}{a_1};$$

$$\therefore \frac{a_{n+1} - a_n}{a_n} = \frac{a_n - a_{n-1}}{a_{n-1}} = \ldots = \frac{a_2 - a_1}{a_1},$$

and therefore these fractions are equal to $\dfrac{a_{n+1} - a_1}{a_n + a_{n-1} + \ldots + a_1}$.

In book IX are also proved the fundamental theorem that a number can be resolved into prime factors uniquely, and the theorem that the number of primes is infinite. IX 36 shows that if $2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is perfect.

To appreciate the remarkable tenth book the reader should, if possible, refer either to Heath's *Euclid*, or to the summary given in volume I of Heath's *Greek Mathematics*. Much of the tenth book is taken up with an investigation of numbers of the form $\sqrt{(\sqrt{a} + \sqrt{b})}$.

The term 'method of exhaustion' is sometimes used of a proof which proceeds by exhausting all other possibilities than the desired one. For example it is sometimes convenient to show that $a > b$ by proving that neither $a = b$ nor $a < b$ can be true. This in fact often amounts to a proof by *reductio ad absurdum*, for it is often shown that $a = b$ and $a < b$ both lead to contradictions. Euclid's method of exhaustion, which is probably due to Eudoxus, has a different meaning. This meaning will be illustrated by his proof of XII 2, which depends upon

> X 1 : Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out.

Here, as in book V, the essence of the method is the use of inequalities. The 'lesser magnitude set out' is the ϵ of modern analysis.

In XII 2 Euclid uses X 1 to deduce that circles are to one another as the squares of their diameters from the fact that similar polygons inscribed in the circles are in that ratio (XII 1).

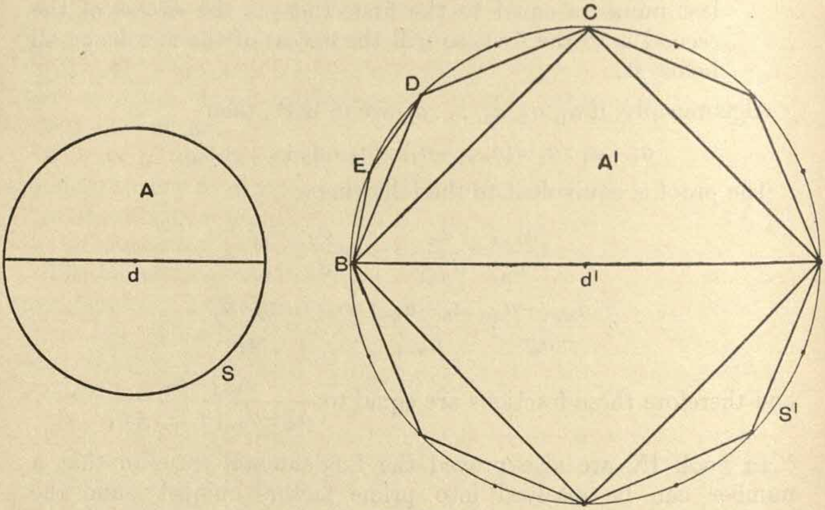


FIG. 131.

If the ratio $A : A'$ of the areas of the circles S, S' is not equal to the ratio $d^2 : d'^2$ of the squares of their diameters, suppose that

$$d^2 : d'^2 = A : X, \text{ where } X < A'.$$

It is easy to show that a square inscribed in S' has area greater than $\frac{1}{2}A'$, and that if the four arcs BC etc. are bisected, the triangles BDC etc. are greater than half the segments in which they are inscribed, and that if the eight arcs BD etc. are bisected, the triangles BED etc. are greater than half the segments in which they are inscribed, and so on.

Thus if from the circle S' there is removed first the square, then the triangles BDC etc., then the triangles BED etc., and so on the conditions of X 1 are satisfied and there will eventually remain an area less than any assigned area.

If P' is the area of the polygon that has been removed from S' when the remainder becomes less than $A'-X$,

then $$A'-P'<A'-X,$$

$$\therefore P'>X, \qquad \text{(i)}$$

$$\therefore A'>X.$$

But if P is the area of a polygon inscribed in S, similar to that removed from S,

$$P:P'=d^2:d'^2=A:X,$$

and $P<A$, $$\therefore P'<X. \qquad \text{(ii)}$$

A contradiction is thus obtained in (i) and (ii). Similarly a contradiction can be derived from the supposition $d^{2\prime}:d'^2=A:Y$ where $Y>A'$. Hence $d^2:d'^2=A:A'$.

In book XI there is a return to geometry proper, but geometry of three dimensions. There are 39 propositions, of which the first half deal with straight lines and planes, parallelism, and perpendicularity; there are also results about solid angles. The second half treats of parallelepipeds, proving for instance

> XI 31 : Parallelepipedal solids which are on equal bases and of the same height are equal to one another,

and including similarity, e.g.

> XI 33 : Similar parallelepipedal solids are to one another in the triplicate ratio of their corresponding sides.

One proposition of book XII has already been quoted. Euclid's method of exhaustion is used in XII 5, 7, 10 to show that a pyramid is a third part of the prism which has the same base and equal height, and to prove the corresponding result for the cone and cylinder. Results of XI are then extended from parallelepipeds to these other solids.

In book XIII there are constructions for the five regular solids and the length of the edge of each is determined in terms of the radius of the circumscribing sphere. This should perhaps be regarded as the ultimate aim of the whole treatise. The construction of the regular pentagon in book IV was a necessary step towards the constructions of book XIII, and some of the investigations of book X are required for the calculation of the edges of the solids.

The chain of constructions already mentioned, ending in VI 25, is of special interest and should be borne in mind by every teacher of geometry. The constructions of VI 28, 29 are for us mostly of historic interest, since the problems to which the Greeks have applied them are handled nowadays by other methods. In V defn. 5

and X 1 we have methods which must always be the foundation of exact analysis, though the language in which they are set out may be unfamiliar.

It is however the logical fashion in which the whole edifice of geometry has been constructed in these thirteen books that chiefly calls for our admiration.

9 THE PRIMITIVE CONCEPTS OF GEOMETRY

The question may be asked: which idea comes first, angle or direction; possibly it is like the question as to the relative priority of the hen and the egg, but the fact that it does arise seems to make it worth while to consider how we get our fundamental geometrical concepts—among which are to be included angle and distance certainly, direction more doubtfully. Children come to school with a fairly adequate notion of distance but are at best extremely hazy about angle; they are accustomed to notice and even to measure the former but not the latter. If we attempt to plunge them into formal geometry without taking steps to develop the missing concept of angle we necessarily get into trouble.

For formal geometry takes both concepts, distance and angle, for granted, does not define them, but lays down postulates to supply the place of definitions. It is the business of the teacher to secure that his pupils shall understand what he is going to talk about before any formal study begins.

Hence it is relevant to enquire how the concepts of distance and angle, and of direction if there is one, do in fact arise or how they can be made to arise in minds in which they are not yet present.

Clearly, like many of our concepts, they arise from our bodily experiences, especially from our movements. These are generally of a complicated character but two simple types can be singled out which, when we have sufficiently refined them, we shall call rotations and translations respectively.

We are quite familiar with the experience of the body as a whole remaining at rest, while the eye alone moves. We say the eye turns and are quite aware of its turning more or less. We say that we look, first in one direction then in another; or that the eye turns from one direction to another; and we can say that one turn is greater or less than another though we can hardly say that two turns are equal from our bodily experience alone.

We cannot turn the eye completely round in its socket; but we can keep the eye fixed in its socket and turn the whole body without changing the spot on which we are standing and so achieve the same object of looking, first in one direction then in another—and now we can turn completely round. We thus get one recognisable, easily identified, amount of turning—the complete turn. If we can, by the help of things external to the body, find satisfactory means of subdividing the complete turn into 'equal' parts we can measure and accurately compare different amounts of turning. Or, what is perhaps better, if we can find means of making 'equal' turns several times in succession we can find what fraction the original turn was of the complete turn.

The idea and the bodily experience involved in this are simple enough, but we cannot execute the necessary operations with any accuracy without external help; we should find it difficult to execute even a complete turn exactly in pitch darkness. For a complete turn a single external mark is sufficient, but for subdivision or repetition we need a series of marks and they must be properly located. This means that for the measurement of angles we are dependent on the measurement of distances, and we cannot achieve our purpose without moving from one spot to another.

The crude measurement of distance, e.g. by pacing,[1] is familiar to all, i.e. the concept of distance is already established. If then we admit the ordinary processes of measuring or comparing distances we can also measure angles and as regards the latter we have the advantage of a natural unit. That is to say angles and distances are measurable quantities independent of time and place—so long as we do not involve ourselves in the difficulties and subtleties of the philosophers and relativists. But what about direction?

We have used the word in a general fashion—we look first in one direction and then in another, the *change* being the really notable thing which we actually experience; but has the term 'direction' any clear meaning if used in isolation, as for instance 'angle' and 'distance' have? It is clearly nothing measurable itself, have we any ground for saying even that two directions are the same?

In common life, so long as we move within a limited space, we certainly do speak of certain standard directions—vertical, north, south, east and west; but how far can this common usage be carried over into scientific geometry? Here there is a distinction which it is vitally important to be clear about and to observe, In the preliminary informal work which is meant to lay the basis for the study of geometry proper, the more nearly we keep to the experience, ideas, and language of common life, the better. Here it is important to avoid confusing children with over-refinements. It is necessary to accustom them to using the ordinary terms, vertical etc., in the ordinary way, but with precision. They must realise for instance that N.W. is exactly half way between N. and W.; that vertical means upright; that ‘perpendicular’ is meaningless without the addition ‘to so and so’. They must recognise that within their ordinary experience ‘west’ always means one definite direction, though they may properly be reminded that west at New York is not the same as west at London. And this natural method of perception and thought may properly be used to make them familiar with the basic propositions as to angles: the sum of the exterior angles of a polygon, the angle-sum of a triangle and the angle properties of parallels.

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[1] More fundamental still is the equation of a distance to the length of some part of our body—a hand's breadth, a cubit, a foot, etc.

These propositions made convincing in this fashion, like those on congruence, may then be properly made the starting point for deduction and development.

But whenever, at a much later stage, if at all, the question arises as to the ultimate bases of the whole geometrical structure, the matter is quite different. We can no longer use any term or concept to which we have not given a perfectly precise form, either by definition or by postulates, and as yet no such precise form has been given to the term direction. Nor can one be given, as we will show. The thing does not exist, as a mathematical entity, unless and until we establish or postulate the doctrine of parallels. In the scientific sense—as distinguished from what is legitimate and useful in the informal preliminary treatment—the doctrine of parallels cannot be based on direction.

The original use of the word is, we have seen, that when we turn the eye we change the direction in which we are looking. We are in the habit of thinking that any particular direction can fitly be expressed by a particular straight line drawn from the eye. Suppose we admit so much and go further, though this may be more questionable, and admit that direction is something which remains unchanged as we pass along the straight line, how much better off shall we be?

The one thing in the original usage of the word which is clear and certain is that any turn of the eye means a change of direction; if then directions are expressed by straight lines, two lines diverging from the same point have different directions. But if two lines converge to a point they also diverge after passing the point; therefore lines which meet have different directions.

So the search for lines with the same direction, if there are such lines, becomes the search for lines which do not intersect. It is easy to imagine and to represent pictorially lines which do not intersect within the limited range accessible to us; but the issue cannot be settled experimentally, for straight lines are not material things on which we can experiment, still less can we carry them on for ever.

We are inevitably thrown back on what may perhaps be called pure speculation. There are three possibilities open *a priori* :

(1) there may be no coplanar lines which do not meet ;
(2) through a point P there may be one and only one line which does not meet a given line a ;
(3) through P there may be more than one line which does not meet a.

Since we cannot find any *logical* ground for rejecting one or other of those three *a priori* possibilities, they all remain open—for, once more, the question is one of concepts, not of percepts, and cannot be settled by experiment.

If we adopt the first of the three possibilities, viz. that all lines meet, it follows that all lines differ from one another in direction.

If we adopt the third it is clear that, as the various lines through P all differ from one another in direction, at the best only one can have the same direction as a ; which of the many?

We might be tempted to choose a particular one which lies symmetrically among the rest : for instance the pencil of lines through P not meeting a may be bounded by two, b and c, and we might choose the bisector of the angle between these.

In fact if we draw the perpendicular from P to a and assume complete symmetry on the two sides of this line, we shall choose the line through P at right angles to this perpendicular.

So far so good : we have chosen a unique line through P as tentatively having the same direction as a. But take another point Q on this line and draw the perpendicular from Q to a ; is this also the line through Q at right angles to the new perpendicular?

If it is not, the line has not at Q the same direction as a ; hence the line has not the same direction at Q as it has at P, which contradicts our previous convention that ‘ direction ’ remains unchanged along a straight line.

If it is, we have a four-sided rectilineal figure with all its angles right angles, and we can easily deduce from this the Euclidean doctrine of parallels ; and this of course means the acceptance of the second of our three alternatives, viz. the supposition that through P there is one and only one line which does not meet a.

It is clear therefore that ‘ direction ’ is not a valid concept unless we adopt the Euclidean axiom of the unique parallel ; therefore it cannot be used as the basis on which to prove that axiom.

10 DEFINITIONS

We do not now trouble beginners much with formal definitions, but the question still crops up from time to time: what is the definition of a straight line?

The proper answer is that there is none; but this paradox needs understanding.

The 'elements' which enter into geometry (point, line, plane, figure, etc.) are not material things, nor the 'percepts' we have of such things, but 'concepts', which, though they may be derived from percepts of the external world, are purely mental constructs. Among the particular percepts, from which the concept of a straight line is derived, are the following:

(1) a string drawn tight between two points;
(2) an axis of rotation;
(3) a ray of light;
(4) the shortest distance between two points.

We test a straight edge by looking along it; this depends on (3). Or by ruling a line with it, turning it over and ruling again; this illustrates (2). The hinges of a door must be properly aligned or they will be strained or torn off; this again illustrates (2).

As regards (1), (2) and (3) the question is always open, how far does the percept under examination conform with the concept? And the answer is always 'imperfectly'. As regards (4) the question is more obscure, for it involves a further concept, distance.

An individual may have formed one or more concepts; if he wishes to discuss or argue about them with any one else it must first be ascertained whether the concepts of the two parties are identical. Even if the individual wishes to discuss his own concepts with himself, to examine their mutual relations or to build a developed system upon them, each concept needs to be clearly particularised.

They thus fall into two classes which may be called primary and secondary. Of these the latter are those which can be defined or described exactly in terms of the former, e.g. a rhombus is a plane figure bounded by four equal sides: such definitions present no difficulty.

But the definition just given involves at least two concepts, viz. 'plane' and 'equal', which themselves require particularising.

Euclid's definition of a plane, like those of point and straight line, is useless—a mere attempt at a description on which no argument can be based. His omission has been made good ; and the ordinary definition of a plane is ' a surface such that, any two points being taken in it, the join of those points lies wholly in the plane '.

That there are such surfaces cannot be proved and their existence has to be taken as axiomatic, but for our present purpose the important thing to notice is that the concept ' plane ' presupposes the concept ' straight '.

The ' straight line ' is in fact one of the primary concepts which cannot be defined in terms of others more fundamental still. Hence no ' definition ' of it is possible.

But before we can argue and develop we must have some properties to start with ; and in the absence of a definition stating such properties we lay down postulates to take its place. Thus the definition of a straight line is contained disguised in the relevant postulates, viz.

1 One and only one straight line can be drawn from any one point to any other ;

2 A straight line can be produced indefinitely in either direction ; and, for the Euclidean straight line,

3 The axiom of parallels or an equivalent.

There is one other percept or matter of experience which is important in this connection.

When a carpenter is ' truing up ' a piece of wood he first planes a ' face ', then an ' edge ' at right angles to the face. He then sets his gauge and runs it along the edge, so marking a line on the face : this line, he takes it for granted, is straight.

The proposition corresponding to this in conceptual geometry is : the locus of points equidistant from a straight line is itself a straight line.

We may go on to add ' parallel to the original straight line ', but from our present point of view the emphasis is on the word ' straight '. The proof that the locus is straight involves the Euclidean axiom of parallels ; in the geometries which discard that axiom the equidistance locus is not straight.

This proposition therefore may be taken either as a logical deduction from the Euclidean axioms or as itself an alternative to the axiom of parallels, from which that can then be derived as a proposition.

There are many other propositions any one of which may similarly be taken as an axiom instead of the axiom of parallels ; for instance :

a quadrilateral exists having all its angles right angles ;
a pair of triangles of different sizes exist having the angles of the one equal to the angles of the other.

See pp. 125-135.

11 CONGRUENCE: GENERAL PRINCIPLES

Congruence of geometrical figures may be considered from different points of view—from two in particular, polar opposites of one another.

There is the view-point of the ordinary man, traveller, carpenter, builder, engineer, architect, whose concern is with material things, to whom a triangle for instance is a material template. At the opposite extreme there is that of the philosopher whose 'mind's aboon the warks o' creeping things' and to whom that which the ordinary man takes for granted is precisely that which is most mysterious. 'Wisdom is justified of all her children'; but their different ways need to be understood and not confused.

Perhaps we may say that the child needs to learn to coordinate the experiences of his senses, vision, touch, motion, while the philosopher is concerned with the fascinating, because baffling, question what is the relation, indeed how can there be a relation, between my mental experiences and the outside world—if there is such a world.

Coming to our particular subject, geometrical congruence, may we say that to a child, geometrical figures are material things or imperfect material representations of things thought of as material; and that the comparison of two figures is the comparison of two objects, either actually material or imagined: a sphere is a ball, a cube a block, a plane a table top, a straight line an edge, and so on. His business in the geometrical comparison of figures is to get clear as to what and how many tests he must apply to satisfy himself that the figures are duplicates; and this, as all teachers of children know, is in itself a task not without difficulty, but very well suited to the immature powers of the child and if suitably handled, congenial to his interests.

The special business of the philosopher on the other hand is the study of origins and criticism of the processes and conclusions of the naïvely working mind. He brings to the surface the many assumptions which the child or the ordinary man makes as a matter of course and without recognition, and asks very awkward questions: What do you mean by saying that a line PQ is equal to a line AB?

Even granting that the statement has a meaning, how do you establish its truth in a particular case? What do you mean by a straight line? How can you justify speaking of moving a line; a line is the join of two points and points have positions only, how then can a line change its position and yet maintain its identity?

The practical conclusion of all this is that in teaching beginners we may properly ignore all that philosophers have to say about the basic propositions on congruence, with one exception: they say that Euclid's proof of the first case of congruence of triangles is a 'tissue of absurdities'. Clearly there can be no obligation to teach this 'proof' if this criticism is justified. The teacher however has quite a different reason for discarding the proof, viz. that to the ordinary child it is ineffective, a stumbling-block and not a foundation-stone. All that he need take from the philosopher is permission to discard what he himself knows to be unserviceable in practice.

It is essential to observe that though teacher and philosopher agree in the practical conclusion to discard Euclid's proof, their reasons for doing so are quite different and are relevant to minds in quite different stages of development. The philosopher's criticism is at bottom, that the 'proof' does not lay bare the assumptions on which it is founded; the teacher's that it is cast in such a form as to make the argument difficult to follow, that the truth set forth can be perceived by a child much more readily if presented in a different way. He will say for instance: If you all draw a line 3″ long as the base of a triangle, an angle of 35° at the left-hand end and make the left-hand side 4″ long, will you all get the same triangle? And he will expect and get the answer 'yes'. He will then formulate the enunciation of the proposition on this basis and not trouble further about 'proof'. He has not met the philosopher's criticism; he has quite properly ignored it as irrelevant.

The corresponding procedure of the mathematician (one of many open to him) is to say that he will take the proposition as an axiom. This is practically what the teacher does, only as he is dealing with children to whom precision of thought is unfamiliar he has to draw out, in some such way as that just indicated, recognition of the fact that the three assigned data give a complete and unambiguous construction. What the mathematician calls an axiom the child has to be led to take as a matter of course. The difficulty to a child is to keep clearly in his consciousness the fact that three things properly chosen determine a triangle; and he needs many simple exercises before he is master of this fact and can apply it easily and correctly. The philosopher's difficulties are of a totally different order and have no relevance to the teacher's proper procedure.

The contrast is closely analogous to that between a child learning to read and a scholar studying the origins of language. The one learns that C A T spells 'cat'; the other enquires perhaps by what positions of tongue and lips the various sounds are produced, why different languages give different names to the same thing, how language originated at all and so on. To each age and each stage of mental development its proper study; milk for babes, strong meat for grown men.

In Stage A then, at all events, a teacher should go on his way quite unperturbed by anxieties raised by the criticisms of Euclid's treatment of congruence detailed in the 1923 Report. His business is to see that his pupils get a thorough grasp of the three fundamental propositions, which they will not do by the formal treatment proper to a more advanced stage of work. Like the propositions on the angle properties of parallel lines, he must get his pupils to regard them as obvious and familiar truths, to know them accurately and to be able to apply them.

Nor is there any need to enquire further about them in the ordinary course up to the School Certificate. Most examining bodies happily have accepted the position and schedule them as propositions not to be set. Even if, under examination pressure or otherwise, a teacher thinks it desirable to accustom his pupils to proofs 'by superposition', he will at worst be inflicting on them an exercise which others think it better to omit. For his pupils will not have reached the stage of the mathematical purist or of the philosopher; his triangles are figures drawn on a piece of paper which can be moved about without change; and as long as he is of that way of thinking—the only way natural to a child—the process is valid for him; the only relevant question is whether he learns anything from it effectively additional to what he has already learnt in another way.

The first possible legitimate occasion for raising the deeper questions and asking what is the logical basis of these things hitherto taken as obvious presents itself, as with the propositions on parallels, when we come to them in the process, recommended in this report, of working backwards. It becomes gradually clear that every proposition when traced backwards depends on either congruence or parallelism or both. The question then arises, are we to leave it at that and use what time we have for further development, or shall we plunge into the difficult question of origins? There can, we suggest, be little doubt that with the ordinary class, under ordinary school conditions, the former is the wiser course. The other is for specialists and even for them may probably far better be left till much later in their career. For not to all, or perhaps even to any large fraction of specialists, will the interest and value of 'axiomatik' be at all comparable with that of advance.

12 GEOMETRY IN SCOTLAND

The teaching of mathematics in Scotland is dominated by the '*Education (Scotland) Note as to Mathematics* (fourth issue). H.M. Stationery Office, price 4d.'

This modifies the 'third issue' and is effective for the Leaving Certificate Examination in 1938 and onwards.

The 'Note' gives various syllabuses :

(A) Syllabus for three year advanced division courses
[the first three post-primary years].

Geometry.

§§ I, II, III, IV cover Euclid I, i.e. the usual course up to Areas and Pythagoras' Theorem.

§ V Circle. § VI Loci and Inequalities.

(B) Syllabus for Leaving Certificate Courses, Lower Standard.

Geometry adds to above :

§ A. Further study of Area, treated geometrically.
§ B. Proportional division of straight lines.

(C) Syllabus for Leaving Certificate Courses, Higher Standard.

Geometry adds to above : Further Similar Figures ; Golden Section ; also Solid Geometry, the theoretical section of which only includes 'knowledge of the facts' so as to prove mensuration formulae ; also Analytical Geometry of the straight line.

(D) Additional Mathematics subjects.

The Important Distinction. The important distinction between the Scottish and English practice is due to the strict interpretation of the dictum that '*The proofs of all the area propositions included must be geometrical ; proofs which depend wholly or partly on mensuration formulae will not be accepted. This applies to all sections of the syllabus*'.

This begins to make a serious difference in the 'Further study of Area', (B) § A above.

Here, for example, what some books give as

illustration of the algebraic formula $(a+b)^2=a^2+2ab+b^2$

must become

proof of the theorem that if C is on AB

square on AB = sq. on AC + sq. of BC + twice rect. AC . CB

[the notation AB^2, rect. AC . BC is tolerated, especially in riders].

In the extension of Pythagoras' theorem the arrangement

$$\begin{aligned} c^2 &= (a-x)^2+p^2 \\ &= a^2-2ax+x^2+p^2 \\ &= a^2-2ax+b^2 \end{aligned}$$

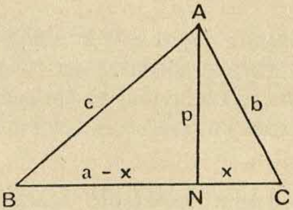

FIG. 132.

would not be tolerated and must become something like :

Since BN is produced to C

the square on BN + twice rect. BC . CN = sq. on BC + sq. on NC.

Add to each the square on AN.

Then since sq. on AB = sum of sqq. on BN, AN,

and sq. on AC = sum of sqq. on CN, AN,

$\therefore$ sq. on AB + twice rect. BC . CN = sum of sqq. on BC, AC.

This geometrical strictness

makes it necessary to prove *Euclid* VI 1 as in *Euclid*,[1] though only for those taking mathematics at the higher standard,

makes necessary great care in proving 'similar triangles are as squares on corresponding sides',

means that a special theorem is necessary to allow the change from $\frac{a}{b}=\frac{c}{d}$ to $ad=bc$,

and involves sundry other similar alterations.

The difficulties due to the existence of incommensurables are discussed in the 1923 Report, pp. 51 to 55, and two alternative methods for the teacher are suggested (p. 53). The former perhaps represents the practice of English teachers. The second involves much less alteration of that practice than would be made if the Scottish practice were adopted.

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[1] Implying knowledge of Eudoxus' def. of proportion $A : B = C : D$ if $mA \gtreqless nB$ according as $mC \gtreqless nD$, m and n being any integers whatever.

Probably all teachers who have considered the matter will have a certain amount of sympathy with the Scottish view and feel that the better pupils lose a good deal by being allowed to assume that all lengths are commensurable. On the other hand few will fail to admit that for the less bright pupils the slacker type of reasoning is much more possible.

13 THE SETTING AND MARKING OF EXAMINATION PAPERS

It seems well to include in this report some suggestions as to the setting and marking of examination papers in geometry. These suggestions deal with a set of individual points of difficulty rather than with general principles, and refer to examinations up to Certificate standard.

Congruence of Triangles. Most examining bodies have now agreed that proof of the three fundamental cases of congruence, being part of the normal basis of assumption, will not now be demanded.

As regards the use of the congruence theorems in other standard theorems or in riders, this report has recommended that the case of congruence used should be named by use of letters *SAS* etc. This is *by no means* to be regarded as a recommendation that examinees should always be penalised for omission to do so. Any deduction of marks for this omission is liable to be unjust or excessive, for the omission is nearly always a failure to adopt the best style rather than a failure of understanding, and moreover the cases differ awkwardly.

Probably no boy who proves two triangles congruent by three sides is ignorant that he is using *SSS*, though he may omit to say so.

If a boy uses *AAS* (or *ASA*) the chance of his not knowing that the side is and must be a ' corresponding ' side is extremely small.

If a boy uses *SAS* and has mentioned ' included ' angle, he has of course made the *SAS* at the end unnecessary ; but even if he has not mentioned ' included ' the chance that he does not know the correct case is small.

If he uses *RHS* without mentioning ' right angle ', or merely putting $\angle A = \angle P$ (right angles) as though a reason for equality, it is possible that he thinks that a pair of equal acute angles would do as well ; but the chances are that the omission is only poor style.

The committee suggest that the omission should involve no penalty in the *SSS* and *AAS* cases, either no penalty or the minimum one in the *SAS* case, and the minimum one in the *RHS* case. They believe that at present the penalty exacted in the *RHS* case is usually much too high. They suggest also that a convenient way of testing a boy's knowledge about the three or four cases of congruence would be to ask directly for an account of them and perhaps also of the cases where equality of three parts fails to ensure congruence.

Standard Theorems with more than One Figure. There are two very important examples of this ;

(i) two parallelograms $ABCD$, $PBCQ$ on same base and between same parallels are equal,
and (ii) the angle AOB at the centre is double the angle at the circumference APB.

Note first that in both these theorems not only two general figures but also a third special figure is sometimes given in text-books, for (i) when D and P coincide, and for (ii) when PB is a diameter.

These *special* figures should *not* be expected unless specially asked for.

Further in (i), if the subtraction method is used, a proof can be given which fits both figures without any change of wording.

In the opinion of the committee a word-perfect proof with only one figure should get full credit.

In (ii) the proof involves ' adding in Fig. (*a*) and subtracting in Fig. (*b*) ' and is incomplete without two figures. Fig. (*b*) is the more difficult case ; but if the correct proof is given for Fig. (*a*), it is fairly certain that it will get 7 or 8 marks out of 10. This leads to teachers telling weak candidates not to bother to learn the hard case.

In view of this, examiners might be well advised to ask for one case or the other, but not for both ; the case of the obtuse angle at the circumference will also be asked for separately.

Theorems with two cases need care from examiners in another way ; for example, it will not do to set without qualification, ' Prove that for the pedal triangle of a triangle ABC the angles are the supplements of $2A$, $2B$, and $2C$, and are bisected internally by the altitudes '.

Miscellaneous. This report echoes the 1923 Report (p. 68) in urging examiners, when setting questions on the tangent theorems or standard locus theorems, to ask for the theorem and its converse if they really want both, or else to expect only the theorem.

Examiners sometimes are kind enough to break up a long proof into several very short pieces ; it needs care when doing this not to introduce the difficulty that candidates are uncertain what amount of proof to give when one step asked for follows immediately from the previous step.

In setting constructions examiners should make it clear whether or not a proof is required. It is frequently desirable to say whether or not the protractor or set-squares may be used, and the words ‘ without calculation ’ may sometimes be added with advantage.

There is no well recognised distinction between the words ‘ draw ’ and ‘ construct ’.

‘ Show how . . . ’ is a phrase which candidates often interpret differently from examiners, e.g. showing something by drawing only when the examiner wishes for a proof.

14 THE RELATIVE IMPORTANCE OF PROPOSITIONS AND RIDERS

It may be argued that this report goes too far in its insistence on learning proofs of standard theorems and on a particular style of writing out these proofs.

If all teachers of geometry were experienced and skilful, the danger would probably not be serious, but many teachers are still so dominated by the ancient tradition that any encouragement that the report might seem to give to the tradition may be harmful.

The traditional opinion has been that propositions are the essential, riders the luxury. This for two reasons :

(i) The propositions are the things of outstanding importance and until they are known, exercises on them are out of place ;
(ii) it is possible even for weaklings to learn proofs, but hardly to do riders, and the only way to make sure of their passing examinations is to concentrate on the former.

Some members of the committee go further than others in challenging this tradition and venture to assert that while the abler pupils can and should master the bookwork, the less able should, at whatever sacrifice of bookwork, occupy themselves mainly or entirely with riders.

That the propositions must be known as facts is obvious : they are the tools with which boys must work. But from this point of view accurate knowledge of the fact is the essential, knowledge of the proof immaterial. In all but extreme cases, of course, the propositions should be proved in class or done by boys individually as exercises, but the practice of going through the proofs again and again in the hope that they will ultimately be mastered is, it is suggested, a waste of time.

By ‘ occupy themselves with riders ’ it is not meant that the class and teacher collectively should spend their whole time doing riders on the blackboard, although much of that work is necessary, but that, just as in algebra and arithmetic boys individually work through sets of examples, so in geometry they should work through sets of riders.

In this process there are various teaching difficulties especially with a full-sized class, and there are also difficulties peculiar to geometry. Insight into geometrical relations commonly comes more slowly than power to execute the comparatively uniform processes of arithmetic. For the geometry teaching, much time is required as well as much discretion on the part of the teacher in giving or withholding help. The guiding principle should be that boys' activity and interest must be secured. This is done more effectively if they are made to do exercises for themselves than if they are merely following and learning ready-made material. Also the mistakes that they make are more natural than those made in reproducing prepared work; they are more interesting and illuminating in themselves and are worthier of criticism and correction. Properly handled, few boys will find this work dull.

It follows, then, that style or lucidity of expression is a power to be gained, often very slowly, by personal effort and practice, not a thing to be imposed. When a boy thinks, or shows in oral discussion, that he has found a solution, he should sometimes be told to express it in writing in his own way: not always, since this takes up too much time. His own expression should be criticised, and he may be asked to improve it, but he should not be required to produce it in a stereotyped form.

The weaker the class the more time will they need for this, and the less time will there be available for drill in the proofs of the standard theorems. Many of these correspond in nature and difficulty with the exercises and will in fact have been included amongst them. But, with the weaker classes, they should remain on a level with the exercises: time will not be available for polishing and memorising the proofs.

The value of mathematics as a training ground in English expression, and on the other hand the necessity for clear expression if there is to be real progress in mathematics, must be constantly kept in view.

A few theorems are best proved by verbal explanations: for example the usual proof that 'angles in the same segment are equal' adds nothing to the statement that they are equal because each of them is half of the angle at the centre which stands on the same arc as they do.

Too many proofs of theorems bear little relation to the kind of work in which the results of the theorems are applied. Euclid's proof of VI 3, beautiful though it is, is not a natural proof; a proof by areas or by the use of sines is more likely to be discovered by the pupil, and this is one of the theorems that should be treated as a rider.

Geometry does not cease to be geometry when use is made of algebra or trigonometry, and the use of these tools should be freely encouraged. It is worth while to teach the cosine and sine formulae

$$a^2=b^2+c^2-2bc\cos A,\quad a/\sin A=b/\sin B=c/\sin C=2R,$$

merely for the purpose of simplifying geometry. Although the trigonometry of the obtuse angle is involved, this is a subject which is well worth teaching.

The variety of methods available is one reason for not insisting on any stereotyped form of proof. Examination requirements may cause difficulty and hesitation. There are cases in which a teacher feels that he is justified in securing a pass at all costs and that he must play for safety even if this means abandoning sound educational methods. Doubts about the soundness of the attitude here adopted towards bookwork may be reinforced by doubts about its practical wisdom in an imperfect world. It must be admitted that no one can count on boys or girls doing riders under examination conditions with the same assurance as he can count on their doing arithmetic. It frequently happens that the slightest hint, possible in class but not in the examination room, makes all the difference between success and total failure. And so the candidate may appear to be far worse than he really is. Against this we can set the even more disturbing fact that in the traditional geometry paper a candidate may scrape through, simply on reproduction, having gained little or nothing by his so-called study of geometry. We believe that the less emphasis is laid on learning and polishing proofs of standard propositions until the power of thinking out proofs has been developed by exercises, the better will it be, not merely for the boy's real development but even for his examination success. Any two-step or three-step theorem should be prefaced by simple riders which illustrate the argument of the theorem; and, after the theorem has been discussed, the ideas involved should be illustrated by further riders. In this way, not only will some of the time spent on the theorem become in effect time spent on rider work, but the theorem itself is likely to be better appreciated by the pupil.

Even so, some modification of the style of certain examinations is highly desirable. There should be a larger proportion of really easy exercises and less drastic criticism of details in the reproduction of bookwork. More questions testing knowledge of the content of the bookwork, i.e. knowledge of the facts as distinct from knowledge of the proofs, would be a great gain.